q-Quaternions and q-deformed $su(2)$ instantons

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Abstract

We construct (anti)instanton solutions of a would-be q-deformed $su(2)$ Yang-Mills theory on the quantum Euclidean space $\mathbb{R}_q^4$ [the $SO_q(4)$-covariant noncommutative space] by reinterpreting the function algebra on the latter as a q-quaternion bialgebra. Since the (anti)selfduality equations are covariant under the quantum group of deformed rotations, translations and scale change, by applying the latter we can generate new solutions from the one centered at the origin and with unit size. We also construct multi-instanton solutions. As they depend on noncommuting parameters playing the roles of 'sizes' and 'coordinates of the centers' of the instantons, this indicates that the moduli space of a complete theory should be a noncommutative manifold. Similarly, gauge transformations should be allowed to depend on additional noncommutative parameters.

1 Introduction

The search for instantonic solutions has become a central point of investigation of Yang-Mills gauge theories on noncommutative manifolds after the discovery [38] that deforming $\mathbb{R}^4$ into the Moyal-Weyl noncommutative Euclidean space regularizes the zero-size singularities of the instanton moduli space (see also [47]). Other noncommutative gemetries have been considered, mostly deformations [11, 5, 12, 31] of the sphere S^4, because the latter, as a compactification of $\mathbb{R}^4$, provides a better framework to display the topological properties of the instanton bundles. It is therefore tempting to investigate this issue also on another available deformation of $\mathbb{R}^4$, the Faddeev-Reshetikhin-Takhtadjan noncommutative Euclidean space $\mathbb{R}^4_q$ covariant under $SO_q(4)$ [15].

At least in our opinion, there is still no fully satisfactory formulation of gauge field theory on quantum group covariant noncommutative spaces (shortly: quantum spaces) like $\mathbb{R}^4_q$ (see e.g. [28] for an attempt). One main reason is the lack of a proper (i.e. cyclic) trace to define gauge-invariant observables (action, etc). Another one is the $\star$-structure of the differential calculus, which for real q is problematic. Probably a satisfactory formulation will be possible within a generalization of the standard framework of noncommutative geometry [9] where gauge transformations, gauge potentials, and the corresponding field strengths will depend not only on coordinate, but also on derivatives (as suggested e.g. in [13, 1]) and/or possibly on additional noncommuting parameters (see section 6 below). Here we leave these issues aside and just ask for nontrivial differential 2-forms solutions of the deformed (anti)selfduality equations: results in this direction might contribute to suggest more general formulations of gauge theories on noncommutative manifolds that include quantum spaces.

As known, the search and classification [3] of Yang-Mills instantons on $\mathbb{R}^4$ is greatly simplified when the latter is endowed with the structure of a quaternion algebra $\mathbb{H}$. Therefore, following the undeformed case, we first (section 2) introduce a notion of a q-quaternion as a 2×2 matrix which can be factorized as the product of the defining matrix of $SU_q(2)$ by an element of a semigroup isomorphic to the semigroup $\mathbb{R}^{\geq}$ of nonnegative real numbers, and reformulate the algebra $\mathcal{A}$ of functions on $\mathbb{R}^4_q$ as a $\star$-bialgebra $C(\mathbb{H}_q)$. The bialgebra structure encodes the property that the product of two quaternions is a quaternion and is inherited from the bialgebra of 2×2 quantum matrices [14, 16, 54, 15] (therefore it differs from the proposal in [36]). We shall give more details and further developments in Ref. [23]. It also turns out that our $\star$-algebra $\mathcal{A}$ and the $C^\star$-algebra of functions on the quantum 4-sphere of Ref. [12] are made to be isomorphic (as $\star$-algebras) if they are slightly extended so as to contain suitable rational functions of their respective central elements; therefore that noncommutative sphere can be regarded as a compactification of $\mathcal{A}$. In section 3 we reformulate in q-quaternion language the $SO_q(4)$-covariant

differential calculus [this turns out to coincide with the bicovariant differential calculus on $M_q(2), GL_q(2)$ [44, 46], and after imposing the unit q-determinant condition with the Woronowicz 4D- bicovariant differential calculus [53, 43] on $C\left(SU_q(2)\right)$], the $SO_q(4)$-covariant q-epsilon tensor and Hodge map [21, 19, 37, 20] on $\Omega^*(\mathbb{R}^4_q)$. In section 4 we recall some basic notions about the standard framework [9] for gauge theories on noncommutative spaces, pointing out where it doesn't fit the present model, and we formulate (anti)selfduality equations. In section 5 we find a large family of solutions A of the (anti)self-duality equations in the form of 1-form valued 2×2 matrices both in the "regular" and in the "singular gauge". There is a larger indeterminacy than in the undeformed theory because we are not yet able to formulate and impose the correct antihermiticity condition on the gauge potential. Among the solutions there are some distinguished choices that closely resemble (in q-quaternion language) their undeformed counterparts (instantons and anti-instantons) in $su(2)$ Yang-Mills theory on $\mathbb{R}^4$. The (still missing) complete gauge theory might however be a deformed $u(2)$ rather than $su(2)$ Yang-Mills theory. We also make contact with the today standard formulation [9] of gauge theory on noncommutative spaces based on the identification of vector bundles on the latter with projective modules over $\mathcal{A}$ by constructing in q-quaternion language the hermitean projector associated to the q-deformed instanton projective module, and we find that it coincides (for a specific choice of the instanton size parameter) with the one found in Ref. [12]. As in the undeformed (and in the Nekrasov-Schwarz [38]) case, applying (section 6) the quantum group $SO_q(4)$ of q-deformed rotations one obtains gauge equivalent solutions (by a global gauge transformation), whereas applying that of q-deformed dilatations and the braided group of q-deformed translations one finds gauge inequivalent solutions. The difference is however that a dependence on additional noncommutative parameters is introduced: this global gauge transformation depends the noncommuting coordinates of $SO_q(4)$, whereas the gauge inequivalent solutions depend on the noncommuting "coordinates of the center" of the (anti)instanton. Finally (section 7), we find first n-instantons solutions in the "singular" gauge for any integer n; the construction procedure is not yet the deformed analog of the general ADHM one[3], but rather of the procedure initiated in [50] and developed in [51], which reduces to the determination of a suitable scalar potential, expressed in quaternion language. Then for $n = 1, 2$ we transform the singular solutions into "regular" solutions by "singular gauge transformations", as in the undeformed case (of course the $n = 1$ regular instanton solution is again the one found in section 4). The solutions are parametrized by noncommuting parameters playing the role of "sizes" and "coordinates of the centers" of the (anti)instantons. This indicates that the moduli space of a complete theory will be a noncommutative manifold.

2 Promoting $C\left(\mathbb{R}_q^4\right)$ to the q-quaternion bi- (or Hopf) algebra $C\left(\mathbb{H}_q\right)$

We start by recalling how the (undeformed) quaternion $\star$-algebra $\mathbb{H}$ can be formulated in terms of 2×2 matrices: any $X \in \mathbb{H}$ is given by

$$X = \mathrm{x}_1 + \mathrm{x}_2 i + \mathrm{x}_3 j + \mathrm{x}_4 k,$$

with $\mathrm{x} \in \mathbb{R}^4$ and imaginary i, j, k fulfilling

$$i^2 = j^2 = k^2 = -1, \qquad\qquad ijk = -1.$$

One refers to x_1 and to the following three terms as to the 'real' and 'imaginary' part of X respectively. Replacing i, j, k by Pauli matrices times the imaginary unit i we can associate to X a matrix

$$X \leftrightarrow x \equiv \begin{pmatrix} \mathrm{x}_1 + \mathrm{x}_4\mathrm{i} & \mathrm{x}_3 + \mathrm{x}_2\mathrm{i} \\ -\mathrm{x}_3 + \mathrm{x}_2\mathrm{i} & \mathrm{x}_1 - \mathrm{x}_4\mathrm{i} \end{pmatrix} =: \begin{pmatrix} \alpha & -\gamma^\star \\ \gamma & \alpha^\star \end{pmatrix}$$

(where $\alpha, \gamma \in \mathbb{C}$). The quaternionic product becomes represented by matrix multiplication, and the quaternionic conjugation becomes represented by hermitean conjugation of the matrix x. Therefore $\mathbb{H}$ can be seen also as the subalgebra of $M_2(\mathbb{C})$ consisting of all complex 2×2 matrices of this form. Since the determinant of any x is nonnegative,

$$|x|^2 \equiv \det(x) = |a|^2 + |\gamma|^2 \geq 0,$$

any x can be factorized in the form

$$x = T|x|,$$

where $T \in SU(2)$ and $|x|$ belongs to the semigroup $\mathbb{R}^{\geq}$ of nonnegative real numbers. Hence any x belongs also to the semigroup $SU(2) \times \mathbb{R}^{\geq}$.

We q-deform this just replacing $SU(2)$ by $SU_q(2)$ in the dual picture of the algebra of functions of the matrix elements of x. In other words, we define a q-quaternion just as one introduces the defining matrix of $SU_q(2)$ [52, 54], but without imposing the unit q-determinant condition. For $q \in \mathbb{R} \setminus \{0\}$ consider the unital associative $\star$-algebra $\mathcal{A} \equiv C(\mathbb{H}_q)$ generated by elements $\alpha, \gamma^\star, \alpha^\star, \gamma$ fulfilling the commutation relations

$$\begin{aligned} &\alpha\gamma = q\gamma\alpha, \qquad \alpha\gamma^\star = q\gamma^\star\alpha, \qquad \gamma\alpha^\star = q\alpha^\star\gamma, \\ &\gamma^\star\alpha^\star = q\alpha^\star\gamma^\star, \qquad [\alpha, \alpha^\star] = (1-q^2)\gamma\gamma^\star \qquad [\gamma^\star, \gamma] = 0. \end{aligned} \tag{2.1}$$

Introducing the matrix

$$x \equiv \begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix} := \begin{pmatrix} \alpha & -q\gamma^\star \\ \gamma & \alpha^\star \end{pmatrix} \tag{2.2}$$

we can rewrite these commutation relations as

$$\hat{R}x_1x_2 = x_1x_2\hat{R} \tag{2.3}$$

and the conjugation relations as $x^{\alpha\beta\star} = \epsilon^{\beta\gamma}x^{\delta\gamma}\epsilon_{\delta\alpha}$, i.e.

$$x^\dagger = \bar{x} \qquad\qquad \text{where } \bar{a} := \epsilon a^T \epsilon^{-1} \quad \forall a \in M_2. \tag{2.4}$$

Here as usual $x_1 \equiv x \otimes_{\mathbb{C}} I_2$, $x_2 \equiv I_2 \otimes_{\mathbb{C}} x$ (I_2 is the 2×2 unit matrix), $\hat{R}$ is the braid matrix of $M_q(2)$, $GL_q(2)$ and $SU_q(2)$

$$\hat{R}^{\alpha\beta}_{\gamma\delta} = q\delta^\alpha_\gamma\delta^\beta_\delta + \epsilon^{\alpha\beta}\epsilon_{\gamma\delta}, \tag{2.5}$$

and ϵ is the corresponding completely q-antisymmetric tensor

$$\epsilon \equiv (\epsilon_{\alpha\beta}) := \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \qquad\qquad \epsilon^{-1} \equiv (\epsilon^{\alpha\beta}) = -q^{-1}(\epsilon_{\alpha\beta}). \tag{2.6}$$

So $\mathcal{A} := C(\mathbb{H}_q)$ can be naturally endowed with a $\star$-bialgebra structure (we are not excluding $\mathbf{0}_2$ from the spectrum of x), more precisely the above real section of the bialgebra $C\left(M_q(2)\right)$ of 2×2 quantum matrices [14, 16, 54, 15]. In the sequel we shall write the corresponding coproduct

$$\Delta(x^{\alpha\gamma}) = x^{\alpha\beta} \otimes x^{\beta\gamma} \tag{2.7}$$

in the more compact matrix product form

$$\Delta(x) = ax \tag{2.8}$$

where we have renamed $x \otimes \mathbf{1} \to a$, $\mathbf{1} \otimes x \to x$. Since the coproduct is a $\star$-algebra map, $\Delta(x)$, or equivalently the matrix product ax of any two matrices a, x with mutually commuting entries and fulfilling (2.3-2.4), again fulfills the latter. Therefore we shall call any such matrix x a *q-quaternion*, and $\mathcal{A} := C(\mathbb{H}_q)$ the q-quaternion bialgebra. Note that, according to this definition, the unit matrix is a q-quaternion. Note that I_2 is a q-quaternion, and x is a q-quaternion iff $-x$ is.

As well-known, the socalled 'q-determinant' of x

$$\begin{aligned} |x|^2 &\equiv \det{}_q(x) := x^{11}x^{22} - qx^{12}x^{21} = \alpha^\star\alpha + \gamma^\star\gamma \\ &= \frac{1}{1+q^2} x^{\alpha\alpha'}x^{\beta\beta'}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}, \end{aligned} \tag{2.9}$$

is central, manifestly nonnegative-definite and group-like. Therefore at representation level it will have zero eigenvalue iff x has $\mathbf{0}_2$ eigenvalue matrix. Replacing (2.5) in (2.3) we find that the latter is equivalent to

$$x\bar{x} = \bar{x}x = |x|^2 I_2. \tag{2.10}$$

If we extend $C(\mathbb{H}_q)$ by the new (central, positive-definite and group-like) generator $|x|^{-1}$ (this will exclude $x = \mathbf{0}_2$ from the spectrum), the

matrix x becomes invertible and we obtain even a Hopf $\star$-algebra with antipode S defined by

$$Sx = x^{-1} = \frac{\bar{x}}{|x|^2}, \qquad\qquad S|x|^{-1} = |x|. \tag{2.11}$$

The matrix elements of $T := \frac{x}{|x|}$ fulfill the 'RTT' [15] relations (2.3) and

$$T^\dagger = T^{-1} = \overline{T}, \qquad\qquad \det{}_q(T) = \mathbf{1}, \tag{2.12}$$

namely generate $C\left(SU_q(2)\right)$ [52, 54] as a quotient subalgebra. Therefore the $x^{\alpha\alpha'}$ generate the (Hopf) $\star$-algebra $C\left(SU_q(2)\times GL^+(1)\right)$ of functions on the "quantum group $SU_q(2)\times GL^+(1)$ of nonvanishing q-quaternions" [a real section of the Hopf algebra $C\left(GL^+_q(2)\right)$], in analogy with the $q=1$ case.

In view of the construction of instanton solutions we also extend $\mathcal{A} = C(\mathbb{H}_q)$ by adding as generators

$$\frac{1}{1+\frac{|x|^2}{\rho^2}}, \qquad\qquad \rho \in \mathbb{R}^+.$$

Identifying $\mathbb{H}_q$ as $\mathbb{R}^4_q$

One can easily verify that as a $\star$-algebra $\mathcal{A} := C(\mathbb{H}_q)$ coincides with the algebra of functions on the $SO_q(4)$-covariant quantum Euclidean Space $\mathbb{R}^4_q$ of [15]. We identify the present $qx^{11}, x^{12}, -qx^{21}, x^{22}$ with the generators x^1, x^2, x^3, x^4 of [15] (in their original indices convention) or with the generators x^{-2}, x^{-1}, x^1, x^2 in the convention of Ref. [39] (which has been heavily used by the author of the present work). We shall denote by $B \equiv (B^a_{\alpha\alpha'})$ this (diagonal and invertible) matrix entering the linear transformation $x^a = B^a_{\alpha\alpha'}x^{\alpha\alpha'}$. The braid matrix of $SO_q(4)$ is obtained as

$$\hat{\mathsf{R}} \equiv (\hat{\mathsf{R}}^{ab}_{cd}) = q^{-1}\mathsf{B}(\hat{R}\otimes_{\mathbb{C}}\hat{R})\mathsf{B}^{-1} \tag{2.13}$$

(recall that the tensor product of two braid matrices is again a braid matrix), where $\mathsf{B}^{ab}_{\alpha\beta\alpha'\beta'} := B^a_{\alpha\alpha'}B^b_{\beta\beta'}$. Its decomposition

$$\hat{\mathsf{R}} = q\mathsf{P}_s - q^{-1}\mathsf{P}_A + q^{-3}\mathsf{P}_t \tag{2.14}$$

in orthogonal projectors follows from that of the braid matrix of $M_q(2), GL_q(2), SU_q(2)$,

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a, \tag{2.15}$$

since $\mathsf{P}:= \mathcal{B}(\mathcal{P}\otimes_{\mathbb{C}} \mathcal{P}')\mathcal{B}^{-1}$ is a projector whenever $\mathcal{P}, \mathcal{P}'$ are*. In fact,

$$\begin{aligned}
&\mathsf{P}_s = \mathcal{B}(\mathcal{P}_s\otimes_{\mathbb{C}}\mathcal{P}_s)\mathcal{B}^{-1}, && \mathsf{P}_t = \mathcal{B}(\mathcal{P}_a\otimes_{\mathbb{C}}\mathcal{P}_a)\mathcal{B}^{-1},\\
&\mathsf{P}_a = \mathcal{B}(\mathcal{P}_s\otimes_{\mathbb{C}}\mathcal{P}_a)\mathcal{B}^{-1}, && \mathsf{P}_{a'} = \mathcal{B}(\mathcal{P}_a\otimes_{\mathbb{C}}\mathcal{P}_s)\mathcal{B}^{-1},\\
&\mathsf{P}_A = \mathsf{P}_a + \mathsf{P}_{a'}.
\end{aligned} \tag{2.17}$$

$\mathcal{P}_s$, $\mathcal{P}_a$, are respectively $GL_q(2)$-covariant deformations of the symmetric and antisymmetric projectors, and have dimension 3,1. They can be expressed in terms of the q-deformed ϵ-tensor by

$$\mathcal{P}_a{}^{\alpha\beta}_{\gamma\delta} = -\frac{\epsilon^{\alpha\beta}\epsilon_{\gamma\delta}}{q+q^{-1}}, \qquad \mathcal{P}_s{}^{\alpha\beta}_{\gamma\delta} = \delta^\alpha_\gamma\delta^\beta_\delta + \frac{\epsilon^{\alpha\beta}\epsilon_{\gamma\delta}}{q+q^{-1}}. \tag{2.18}$$

P_s, P_A, P_t are $SO_q(4)$-covariant deformations of the symmetric trace-free, antisymmetric and trace projectors respectively; as we shall see $\mathsf{P}_a, \mathsf{P}_{a'}$ are projectors respectively on the selfdual and antiselfdual 2-forms subspaces. By (2.17) $\mathsf{P}_s, \mathsf{P}_a, \mathsf{P}_{a'}, \mathsf{P}_A, \mathsf{P}_t$ respectively have dimensions 9,3,3,6,1, and

$$\mathsf{P}_t{}^{ij}_{kl} = (g^{sm}g_{sm})^{-1}g^{ij}g_{kl} = \frac{1}{(q+q^{-1})^2}g^{ij}g_{kl} \tag{2.19}$$

where the 4×4 matrix g_{ab} (denoted as C_{ab} in [15]) is given by

$$g_{ab} = B^{-1}{}^{\alpha\alpha'}_{a}B^{-1}{}^{\beta\beta'}_{b}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}; \tag{2.20}$$

it is the $SO_q(4)$-isotropic 2-tensor, deformation of the ordinary Euclidean metric, and "Killing form" of $U_qso(4)$. Recalling that $\hat{R}^T = \hat{R}$ one immediately checks that the commutation relations (2.3) become

$$\mathsf{P}_A{}^{ij}_{kl}\, x^k x^l = 0 \tag{2.21}$$

as in the definition [15] of the quantum Eulidean space.

The commutation relations and the $\star$-structure are covariant under, i.e. preserved by, matrix multiplication

$$x \to a\, x\, b$$

by the defining matrices a, b of two copies $SU_q(2)$, $SU_q(2)'$ of the special unitary quantum group, or of two copies $\mathbb{H}_q$, $\mathbb{H}_q'$ of the quaternion quantum group, respectively, whose entries commute with each other and with the entries of x. In other words they are covariant under the (mixed left-right)

*The orthonormality relations for the $\mathcal{P}_\mu$, with $\mu = s, a$,

$$\mathcal{P}_\mu\mathcal{P}_\nu = \mathcal{P}_\mu\delta_{\mu\nu}, \qquad \sum_\mu \mathcal{P}_\mu = I, \tag{2.16}$$

trivially imply the orthogonality relations for the P_μ, with $\mu = s, a, a', t$.

coactions of $SU_q(2) \otimes SU_q(2)' = Spin_q(4)$ and of $\mathbb{H}_q \otimes \mathbb{H}'_q$. This follows from the fact that the twofold coproduct $\Delta^{(2)}(x) = axb$,

$$\Delta^{(2)}(x^{\alpha\alpha'}) = a^{\alpha\beta}b^{\beta'\alpha'} \otimes x^{\beta\beta'}, \qquad \text{i.e.} \quad x \xrightarrow{\Delta_L} a\,x\,b, \tag{2.22}$$

is a $\star$-homomorphism, or equivalently both the the left coaction $x \rightarrow a\,x$ and the right one $x \rightarrow x\,b$ are.

Upon applying the linear transformation Ḅ (2.22) takes the form

$$\Delta_L(x^i) = \mathbf{T}^i_j \otimes x^j, \qquad \mathbf{T}^i_j := B^i_{\alpha\alpha'}a^{\alpha\beta}b^{\beta'\alpha'}B^{-1}{}^{\beta\beta'}_j. \tag{2.23}$$

Note that the $\mathbf{T}^i_j$ are invariant under the $\mathbb{Z}_2$ action defined by the change of signs $(a,b) \rightarrow (-a,-b)$. Relation $(2.23)_1$ has the same form as the left coaction of Ref. [15] of the quantum group $SO_q(4)$ [and of its extension $\widetilde{SO_q}(4) := SO_q(4) \times GL^+(1)$, the quantum group of rotations and scale transformations in 4 dimensions] on $\mathbb{R}^4_q$. This no formal coincidence: the $\mathbf{T}^i_j$ fulfill the commutation and $\star$-conjugation relations

$$\hat{\mathsf{R}}\,\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_1\mathbf{T}_2\hat{\mathsf{R}}\,, \qquad \mathbf{T}^{i\,\star}_j = g^{jj'}\mathbf{T}^{i'}_{j'}g_{i'i} \tag{2.24}$$

and in addition $g_{ii'}\mathbf{T}^i_j\mathbf{T}^{i'}_{j'} = g_{jj'}\mathbf{1}$ if the central element $|a||b|$ is $1^\dagger$. These are respectively the defining relations of $\widetilde{SO_q}(4)$ and of the compact quantum subgroup $SO_q(4)$ [15]. We have thus an explicit realization of the equivalences

$$SO_q(4) = SU_q(2) \times SU_q(2)'/\mathbb{Z}_2, \qquad \widetilde{SO_q}(4) = \mathbb{H}_q \times \mathbb{H}'_q/GL(1).$$

As we shall recall in section 6, the commutation relations are also invariant under the braided group of translations [33, 34] $\mathbb{R}^4_q$, which is the q-deformed version of the group of translations $\mathbb{R}^4$; the role of composition of translations is played by the socalled braided coaddition. They are in fact covariant under the coaction of the full inhomogenous extension $\widetilde{ISO_q}(4)$ [45] of $\widetilde{SO_q}(4)$ (or quantum Euclidean group in 4 dimensions), which includes q-deformed translations together with scale changes and rotations ($\widetilde{ISO_q}(4)$ can be obtained also by "bosonization" of $\mathbb{R}^4_q$ [33]).

The analogy with the case $q = 1$ would be complete if one were able to further extend the action of $\widetilde{ISO_q}(4)$ into that of a quantum conformal group. This is out of the scope of this work and will hopefully treated elsewhere [23]. A quantum deformation of the Universal Enveloping Algebra (U.E.A.) of the conformal group having the U.E.A. of the q-deformed Poincaré group [40] as a closed subalgebra was already constructed in [29].

†Relation $(2.24)_1$ follows from (2.13) and (2.3) for both a and b^T [note that the transpose b^T also fulfills (2.3), as $\hat{R}^T = \hat{R}$]; relation $(2.24)_2$ follows from (2.4) for both a and b; $g_{ii'}\mathbf{T}^i_j\mathbf{T}^{i'}_{j'} = g_{jj'}$ follows from (2.9), (2.20) when $|a||b| = 1$.

Comparison and links with other formulations

A matrix version of the 4-dim quantum Euclidean space (with no interpretation in terms of q-deformed quaternions) was proposed also in [36]. However, the $\star$-relations and the $SO_q(4)$-coaction are different, i.e. cannot be put both in the form (2.1), (2.22), even by a relabelling of the generators.

As a $\star$-algebra, our $\mathcal{A}$ slightly differs from the one of the quantum 4-sphere S_q^4 proposed in [12] (which was introduced as a 'suspension' of the algebra of the quantum 3-sphere S_q^3), in the sense that a slight extension $\mathcal{A}^{ext}$ of $\mathcal{A}$ by some rational functions of $|x|$ contains that algebra as a $\star$-subalgebra. Define

$$\begin{aligned}
\alpha' &= \sqrt{2}\alpha^\star \frac{2}{1+2|x|^2} e^{ia}, & \alpha'^\star &= \sqrt{2}\alpha \frac{2}{1+2|x|^2} e^{-ia}, \\
\beta' &= \sqrt{2}\gamma^\star \frac{2}{1+2|x|^2} e^{ib}, & \beta'^\star &= \sqrt{2}\gamma \frac{2}{1+2|x|^2} e^{-ib}, \\
z &= \frac{1-2|x|^2}{1+2|x|^2}
\end{aligned} \tag{2.25}$$

where $\alpha, \gamma, \alpha^\star, \gamma^\star$ fulfill (2.1) and $e^{ia}, e^{ib} \in U(1)$ are possible phase factors. Then α', β', z fulfill the defining relation (1) of the $C^\star$-algebra considered in Ref. [12] (where these elements are respectively denoted as α, β, z), in particular

$$\alpha'\alpha'^\star + \beta'\beta'^\star + z^2 = \mathbf{1}, \tag{2.26}$$

and the invertible function $z(|x|)$ spans $[-1, 1[$, i.e. all the spectrum of z except the eigenvalue $z = 1$, as $|x|$ spans all its spectrum $[0, \infty[$. Viceversa, starting from the latter and enlarging it so that it contains the element $(1+z)/2(1-z) =: |x|^2$ then inverting the above formulae one obtains elements $\alpha, \gamma, \alpha^\star\gamma^\star$ fulfilling our defining relations (2.1).

The redefinitions (2.25) have exactly the form of a stereographic projection of $\mathbb{R}^4$ on a sphere S^4 of unit radius (recall that $x \cdot x = 2|x|^2$): S^4 is the sphere centered at the origin and $\mathbb{R}^4$ the subspace $z = 0$ immersing both in a $\mathbb{R}^5$ with coordinates defined by $X \equiv (Re(\alpha'), Im(\alpha'), Re(\beta'), Im(\beta'), z)$. In the commutative theory adjoining the missing point $X = (0, 0, 0, 0, 1)$ of S^4 amounts to adding to $\mathbb{R}^4$ the point at infinity, i.e. to compactifying $\mathbb{R}^4$ to S^4. We can thus regard the transition from our algebra to the one considered in Ref. [12] as a compactification of $\mathbb{R}_q^4$ into their S_q^4.

3 The $SO_q(4)$-covariant differential calculi

The $SO_q(4)$-covariant differential calculus [6] (d, Ω^*) on $\mathbb{R}_q^4 \sim \mathbb{H}_q$ is obtained imposing covariant homogeneous bilinear commutation relations (3.1) between the x^i and the differentials $\xi^i := dx^i$. Partial derivatives

are introduced through the decomposition $d = \xi^a \partial_a = \xi^{\alpha\alpha'}\partial_{\alpha\alpha'}$ of the ($SO_q(4)$-invariant) exterior derivative. All other commutation relations are derived by consistency with nilpotency and the Leibniz rule. Beside (2.21), or equivalently (2.3), we have

$$x^h\xi^i = q\hat{\mathsf{R}}^{hi}_{jk}\xi^j x^k \qquad \Leftrightarrow \qquad x^{\alpha\alpha'}\xi^{\beta\beta'} = \hat{R}^{\alpha\beta}_{\gamma\delta}\hat{R}^{\alpha'\beta'}_{\gamma'\delta'}\xi^{\gamma\gamma'}x^{\delta\delta'}, \tag{3.1}$$

$$(\mathsf{P}_s+\mathsf{P}_t)^{ij}_{hk}\xi^h\xi^k = 0 \qquad \Leftrightarrow \qquad \mathcal{P}_{s\gamma\delta}^{\alpha\beta}\mathcal{P}_{s\gamma'\delta'}^{\alpha'\beta'}\xi^{\gamma\gamma'}\xi^{\delta\delta'} = 0 = (\xi\epsilon\xi^T)^{\gamma\delta}\epsilon_{\gamma\delta}, \tag{3.2}$$

$$\mathsf{P}_{A}{}^{ij}_{hk}\partial_j\partial_i = 0 \qquad \Leftrightarrow \qquad \partial_{\alpha\alpha'}\partial_{\beta\beta'} = \hat{R}^{\delta\gamma}_{\beta\alpha}\hat{R}^{-1}{}^{\delta'\gamma'}_{\beta'\alpha'}\partial_{\gamma\gamma'}\partial_{\delta\delta'}, \tag{3.3}$$

$$\partial_i x^j = \delta^j_i + q\hat{\mathsf{R}}^{jh}_{ik}x^k\partial_h \qquad \Leftrightarrow \qquad \partial_{\alpha\alpha'}x^{\beta\beta'} = \delta^\beta_\alpha\delta^{\beta'}_{\alpha'} + \hat{R}^{\beta\delta}_{\alpha\gamma}\hat{R}^{\beta'\delta'}_{\alpha'\gamma'}x^{\gamma\gamma'}\partial_{\delta\delta'}, \tag{3.4}$$

$$\partial^h\xi^i = q^{-1}\hat{\mathsf{R}}^{hi}_{jk}\xi^j\partial^k \qquad \Leftrightarrow \qquad \partial_{\alpha\alpha'}\xi^{\beta\beta'} = \hat{R}^{-1}{}^{\beta\delta}_{\alpha\gamma}\hat{R}^{-1}{}^{\beta'\delta'}_{\alpha'\gamma'}\xi^{\gamma\gamma'}\partial_{\delta\delta'}. \tag{3.5}$$

[An alternative $SO_q(4)$-covariant differential calculus $(\hat{d}, \hat{\Omega}^*)$ is obtained replacing $q, \hat{\mathsf{R}}$ by $q^{-1}, \hat{\mathsf{R}}^{-1}$ in (3.1-3.5)]. The ξ^i transform under $SO_q(4)$ exactly as the x^i, the ∂_i in the contragredient corepresentation. We introduce the notation

$$\partial^{\alpha\alpha'} := \epsilon^{\alpha\beta}\epsilon^{\alpha'\beta'}\partial_{\beta\beta'}, \qquad\qquad \partial \equiv (\partial^{\alpha\alpha'}). \tag{3.6}$$

The $\partial^{\alpha\alpha'}$ fulfill the same commutation relations (among themselves) as the $x^{\alpha\alpha'}$, and transform in the same way under the $SO_q(4)$ coaction (equivalently, the $\partial^a := g^{ab}\partial_b$ commute and transform as the x^a). As a consequence, the Laplacian $\Box := g^{hk}\partial_k\partial_h = \epsilon^{\alpha\beta}\epsilon^{\alpha'\beta'}\partial_{\beta\beta'}\partial_{\alpha\alpha'}$ is $SO_q(4)$-invariant and commutes with the $\partial_{\alpha\alpha'}$, and

$$\partial\bar{\partial} = \bar{\partial}\partial = I_2|\partial|^2 \equiv I_2\frac{1}{1+q^2}\Box. \tag{3.7}$$

From (3.4), (3.5) it follows

$$|\partial|^2 x = q^{-2}\partial + q^2 x|\partial|^2 \qquad\qquad |\partial|^2\,\xi = q^{-2}\xi\,|\partial|^2 \tag{3.8}$$

$$\partial|x|^2 = q^{-2}x + q^2|x|^2\partial \qquad\qquad \partial\frac{1}{|x|^2} = -q^{-4}\frac{x}{|x|^4} + q^{-2}\frac{1}{|x|^2}\partial \tag{3.9}$$

$$|\partial|^2\,\frac{1}{|x|^2} = \frac{q^{-4}}{|x|^2}|\partial|^2 - \frac{q^{-6}}{|x|^4}x\cdot\partial \tag{3.10}$$

Since the rhs of the latter formula applied to $\mathbf{1}$ gives zero, $1/|x|^2$ is harmonic, as in the undeformed case. There exists a special combination V of $\mathbf{1}, x\cdot\partial, \Box$ which is unitary and fulfills

$$Vx^i = qx^iV, \qquad V\partial^i = q^{-1}\partial^iV, \qquad V\xi^i = \xi^iV.$$

We add as new generator its "inverse square root", a unitary element λ such that $\lambda^2V = V\lambda^2 = \mathbf{1}$ and

$$\lambda x^i = q^{-1}x^i\lambda, \qquad \lambda\partial^i = q\partial^i\lambda, \qquad \lambda\xi^i = \xi^i\lambda. \tag{3.11}$$

We introduce the following unital associative algebras:

- We shall denote by $\bigwedge^*$ (exterior algebra, or algebra of exterior forms) the $\natural$-graded algebra generated by the ξ^i, where the grading $\natural$ is the degree in ξ^i; any component $\bigwedge^p$ having $\natural = p$ carries a corepresentation of $SO_q(4)$ and has the same dimension $\binom{4}{p}$ as in the $q = 1$ case. In particular, up to a factor there exists a unique 4-form which we shall denote as d^4x. $\bigwedge^p$ is irreducible if $p \neq 2$, and, as we shall see, splits into a selfdual and an antiselfdual part if $p = 2$, exactly as in the $q = 1$ case.
- We shall denote by $\mathcal{DC}^*$ ("differential calculus algebra") the $\natural$-graded algebra generated by x^i, ξ^i, ∂_i. Elements of $\mathcal{DC}^p$ are differential-operator-valued p-forms.
- We shall denote by $\Omega^* \equiv$ (algebra of differential forms) the $\natural$-graded subalgebra generated by the ξ^i, x^i. By definition $\Omega^0 = \mathcal{A}$ itself, and both Ω^* and Ω^p are $\mathcal{A}$-bimodules. Also, we shall denote by Ω^*_S the subalgebra and $C(SU_q(2))$ -bimodule generated by $T^{\alpha\alpha'}$, $dT^{\alpha\alpha'}$ (this is still 4-dim!), and by $\tilde{\Omega}^*$ the extension of Ω^* with the unitary generators $\lambda^{\pm 1}$ obeying (3.11).
- We shall denote by $\mathcal{H}$ (Heisenberg algebra) the subalgebra generated by the x^i, ∂_i. By definition, $\mathcal{DC}^0 = \mathcal{H}$, and both $\mathcal{DC}^*$ and $\mathcal{DC}^p$ are $\mathcal{H}$-bimodules.

Remark 1. The whole set of commutation relations (2.3, (3.1-3.5) is [7] in fact invariant under the replacement $x^{\alpha\alpha'}/|x|^2q^2(1-q^2) \to \partial^{\alpha\alpha'}$ (this is an algebra homomorphism).

As a corollary, on Ω^* one can realize the action of the exterior derivative as the (graded) commutator

$$d\omega_p = [-\theta, \omega_p\} := -\theta\omega_p + (-)^p\omega_p\theta, \qquad \omega_p \in \Omega^p \tag{3.12}$$

with the special $SO_q(4)$-invariant 1-form [8, 48] (the 'Dirac Operator', in Connes' [9] parlance)

$$\theta := (d|x|^2)|x|^{-2}\frac{1}{q^2-1} = \frac{q^{-2}}{q^2-1}\xi^{\alpha\alpha'}\frac{x^{\beta\beta'}}{|x|^2}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}. \tag{3.13}$$

θ is closed:

$$d\theta = 0, \qquad \theta^2 = 0. \tag{3.14}$$

Applying d to (2.10) we find

$$x\bar{\xi} + \xi\bar{x} = (q^2-1)\theta|x|^2 I_2, \qquad \bar{x}\xi + \bar{\xi}x = (q^2-1)\theta|x|^2 I_2. \tag{3.15}$$

Relation (3.1) implies $|x|^2\xi^i = q^2\xi^i|x|^2$, which we generalize as usual to

$$|x|\xi^i = q\xi^i|x|, \qquad \Rightarrow \qquad |x|\,\theta = q\,\theta\,|x|. \tag{3.16}$$

By a straightforward computation one also finds

$$dT^{\alpha\alpha'} = q^{-1}\xi^{\alpha\alpha'}\frac{1}{|x|} + (q^{-1}-1)\theta T^{\alpha\alpha'}. \tag{3.17}$$

By (2.22) the 1-form-valued 2×2 matrices

$$(dT)\overline{T}, \qquad (d\overline{T})T \tag{3.18}$$

are manifestly invariant under respectively the right and left coaction of the Hopf algebra $SU_q(2)$, or equivalently the $SU_q(2)'$ and the $SU_q(2)$ part of $SO_q(4)$ coaction. Setting $Q := -\epsilon^{-1}\epsilon^T$ one finds

$$\mathrm{tr}[Q(dT)\overline{T}] = \mathrm{tr}[Q^{-1}(d\overline{T})T] = (q-1)(q-q^{-2})\theta;$$

only in the $q \to 1$ limit these traces vanish. That's why for generic $q \neq 1$ the four matrix elements of either $(dT)\overline{T}$ or $(d\overline{T})T$ are independent, and make up alternative bases for both Ω^*_S and Ω^*.

Actually, one can check (we will give details in [18]) that (d, Ω^*) coincides with the bicovariant differential calculus on $M_q(2), GL_q(2)$ [44, 46], and (d, Ω^*_S) coincides with the Woronowicz 4D- bicovariant one [53, 43] on $C(SU_q(2))$.

One major problem in the present $q \in \mathbb{R} \setminus \{0\}$ case is that the calculus is not real: there is no $\star$-structure such that $d(f^\star) = (df)^\star$, nor is there a $\star$-structure $\star : \Omega^* \to \Omega^*$. Formally, a $\star$-structure would map the commutation relations of (d, Ω^*) into the ones of $(\hat{d}, \hat{\Omega}^*)$, and conversely. At least, there is a $\star$-structure [41]

$$\star : \mathcal{DC}^* \to \mathcal{DC}^*$$

having the desired commutative limit (the $\star$-structure of the De Rham calculus on $\mathbb{R}^4$), but a rather nonlinear character (incidentally, the latter has been recently [17] recast in a much more suggestive form), in other words objects of the second calculus can be realized nonlinearly in terms of objects of the first (and conversely).

One could introduce a simpler [$SO_q(4)$-covariant] $\star$-structure

$$\star" : \tilde{\Omega}^* \to \tilde{\Omega}^*. \tag{3.19}$$

It would be compactly summarized in the formula

$$\theta^{\star"} = -q\lambda^{-2}\theta, \tag{3.20}$$

and would coincide with the one suggested as a side-remark in formula (7.2) of [24]. But this would not be useful for the present purposes, because its $q \to 1$ limit is not the $\star$-structure of the De Rham calculus on $\mathbb{R}^4$[‡], unless in the commutative limit some coordinates x^a vanish (instead of becoming cartesian coordinates).

[‡]Eq. (3.20) is equivalent to $(d|x|^2)^{\star"} = -q^{-1}\lambda^{-2}d|x|^2$. In the limit $q \to 1$ $\lambda \to \mathbf{1}$, so that $(d|x|^2)^{\star"} = -d|x|^2$, i.e. $d|x|^2$ is *purely imaginary*, rather than real! A short computation also shows that in this limit $(\xi\bar{\xi})^{\dagger"} \propto T(\bar{\xi}\xi)T \in \Omega^{2\prime}$, in other words $\star"$ maps selfdual into antiselfdual 2-forms (and conversely), instead of preserving the chirality!

3.1 Hodge operator and (anti)selfdual 2-forms

The **Hodge map** is a $SO_q(4)$-covariant, $\mathcal{A}$-bilinear map $*:\tilde{\Omega}^p \to \tilde{\Omega}^{4-p}$ such that $*^2 = \mathrm{id}$, defined by

$$^*(\xi^{i_1}...\xi^{i_p}) = c_p\, \xi^{i_{p+1}}...\xi^{i_4} \varepsilon_{i_4...i_{p+1}}{}^{i_1...i_p} \lambda^{2p-4},$$

where $\varepsilon^{hijk} \equiv q$-epsilon tensor [21, 19, 37, 20] and c_p are suitable normalization factors [21, 19, 37, 20]. Actually this extends [20] to a $\mathcal{H}$-bilinear map $*:\mathcal{DC}^p \to \mathcal{DC}^{4-p}$ with the same features. For $p=2$ λ-powers disappear and one even gets a map $*:\Omega^2 \to \Omega^2$ defined by

$$^*\xi^i\xi^j = \frac{1}{[2]_q}\xi^h\xi^k\varepsilon_{kh}{}^{ij}\omega_{ji}, \tag{3.21}$$

where $[2]_q = q+q^{-1}$. By an explicit calculation one finds that this amounts to

$$^*\xi^i\xi^j = (\mathsf{P}_a - \mathsf{P}_{a'})^{ij}_{hk}\,\xi^h\xi^k, \tag{3.22}$$

with $\mathsf{P}_a, \mathsf{P}_{a'}$ defined in (2.17). $\bigwedge^2$ splits into the direct sum

$$\bigwedge^2 = \check{\bigwedge}^2 \oplus \check{\bigwedge}^2{}' = \mathsf{P}_a\bigwedge^2 \oplus \mathsf{P}_{a'}\bigwedge^2 \tag{3.23}$$

of the eigenspaces $\check{\bigwedge}^2, \check{\bigwedge}^2{}'$ of $*$ with eigenvalues $1,-1$ (the "subspaces of selfdual and antiselfdual exterior forms" respectively), which carry the (3,1)- and (1,3)-dimensional corepresentation of $SU_q(2)\times SU'_q(2)$. By (2.17), (3.2) and (2.18) $\check{\bigwedge}^2, \check{\bigwedge}^2{}'$ are respectively spanned by

$$f^{\alpha\beta} := \mathcal{P}_{s\,\gamma\delta}^{\alpha\beta}\epsilon_{\gamma'\delta'}\xi^{\gamma\gamma'}\xi^{\delta\delta'} = \epsilon_{\gamma'\delta'}\xi^{\alpha\gamma'}\xi^{\beta\delta'} = (\xi\epsilon\xi^T)^{\alpha\beta} \tag{3.24}$$

and their antiselfdual partner

$$f'^{\alpha'\beta'} := \mathcal{P}_{s\,\gamma'\delta'}^{\alpha'\beta'}\epsilon_{\gamma\delta}\xi^{\gamma\gamma'}\xi^{\delta\delta'} = \epsilon_{\alpha\beta}\xi^{\alpha\alpha'}\xi^{\beta\beta'} = (\xi^T\epsilon\xi)^{\alpha'\beta'}. \tag{3.24$'$}$$

As expected, only three out of the four matrix elements $f^{\alpha\beta}$ (resp. $f'^{\alpha'\beta'}$) are independent, as (3.2) implies $\epsilon_{\alpha\beta}f^{\alpha\beta} = 0 = \epsilon_{\alpha'\beta'}f'^{\alpha'\beta'}$. As a basis we can alternatively use also the matrix elements of $\xi\bar{\xi}$ (resp. $\bar{\xi}\xi$), because

$$(\xi\bar{\xi})^{\alpha\beta} = f^{\alpha\gamma}\epsilon^{\gamma\beta}, \qquad (\bar{\xi}\xi)^{\alpha'\beta'} = \epsilon^{\alpha'\gamma'}f'^{\gamma'\beta'}. \tag{3.25}$$

From the decomposition $\mathsf{P}_A = \mathsf{P}_a + \mathsf{P}_{a'}$ one easily finds

$$\xi^{\alpha\alpha'}\xi^{\beta\beta'} = -\frac{1}{q+q^{-1}}[f^{\alpha\beta}\epsilon^{\alpha'\beta'} + \epsilon^{\alpha\beta}f'^{\alpha'\beta'}] \tag{3.26}$$

Using relations (3.2) and (2.5) one easily derives the following relations

$$x^{\alpha\alpha'}f^{\beta\gamma} = q(\hat{R}_{12}\hat{R}_{23})^{\alpha\beta\gamma}_{\lambda\mu\nu}f^{\lambda\mu}x^{\nu\alpha'}, \tag{3.27}$$

$$\partial^{\alpha\alpha'}f^{\beta\gamma} = q^{-1}(\hat{R}_{12}\hat{R}_{23})^{\alpha\beta\gamma}_{\lambda\mu\nu}f^{\lambda\mu}\partial^{\nu\alpha'}. \tag{3.28}$$

The second is obtained from the first by applying $\Box$ and recalling (3.8) (or, alternatively, Remark 1). As done in $(3.24)'$, in the sequel we shall usually label a formula regarding antiselfdual 2-forms by adding a prime to the label of its selfdual counterpart, and possibly omit it, whenever it can be obtained from the latter by the obvious replacements. As another example, the analog of (3.27) reads

$$x^{\alpha\alpha'} f'^{\beta'\gamma'} = q(\hat{R}_{12}\hat{R}_{23})^{\alpha'\beta'\gamma'}_{\lambda'\mu'\nu'} f'^{\lambda'\mu'} x^{\alpha\nu'}. \tag{3.27$'$}$$

From the previous three formulae and $(3.28)'$ it follows that Ω^2 (resp. $\mathcal{DC}^2$) splits into the direct sum of $\mathcal{A}$- (resp. $\mathcal{H}$-) bimodules

$$\Omega^2 = \check{\Omega}^2 \oplus \check{\Omega}^{2\prime} \qquad\qquad (\text{resp. } \mathcal{DC}^2 = \check{\mathcal{DC}}^2 \oplus \check{\mathcal{DC}}^{2\prime}) \tag{3.29}$$

of the eigenspaces of $*$ with eigenvalues $1, -1$ respectively, whose elements we shall call as usual "self-dual and anti-self-dual 2-forms".

Proposition 1 *For any $\omega_2 \in \check{\Omega}^2$, $\omega_2' \in \check{\Omega}^{2\prime}$, (resp. $\omega_2 \in \check{\mathcal{DC}}^2$, $\omega_2' \in \check{\mathcal{DC}}^{2\prime}$)*

$$\omega_2\, \omega_2' = \omega_2'\, \omega_2 = 0, \tag{3.30}$$

Proof Since $\check{\Omega}^2, \check{\Omega}^{2\prime}$ are $\mathcal{A}$-bimodules (resp. $\check{\mathcal{DC}}^2, \check{\mathcal{DC}}^{2\prime}$ are $\mathcal{H}$-bimodules) to prove (3.30) it is sufficient to prove

$$f^{\alpha\beta} f'^{\gamma'\delta'} = 0, \qquad\qquad f'^{\gamma'\delta'} f^{\alpha\beta} = 0.$$

By construction the lhs's belong to the $(3,3)$-dimensional (irreducible) corepresentation of $SU_q(2) \times SU_q'(2)$; at the same time, being 4-forms, they must be proportional to the invariant 4-form d^4x, i.e. belong to the $(1,1)$-dimensional corepresentation. Therefore they have to vanish. $\Box$

The 2-forms $(\xi\bar{\xi})^{\alpha\beta}$, $(\bar{\xi}\xi)^{\alpha'\beta'}$ are exact. One can find 1-form-valued matrices a, a' such that

$$d\, a = \xi\bar{\xi}, \qquad\qquad d\, a' = \bar{\xi}\xi. \tag{3.31}$$

Clearly, they are defined up to d-exact terms. Among the simplest choices we have

$$\hat{a} := -\xi\bar{x}, \qquad\qquad \hat{a}' := -\bar{\xi}x. \tag{3.32}$$

They have the following commutation relations with the coordinates:

$$x^{\alpha\alpha'}(\hat{a}\epsilon)^{\beta\gamma} = (q\hat{R}_{12}\hat{R}_{23}^{-1})^{\alpha\beta\gamma}_{\lambda\mu\nu}(\hat{a}\epsilon)^{\lambda\mu} x^{\nu\alpha'}$$

(and similarly for $\hat{a}'$). The four matrix elements of $\hat{a}$ are all independent and make up an alternative basis for Ω^1; they belong to the $(3,1) \oplus (1,1)$-dimensional (reducible) corepresentation of $SU_q(2) \times SU_q'(2)$. (And similarly for $\hat{a}'$). These properties remain true for any combination

$$a_\kappa := \hat{a} + \kappa\, \theta |x|^2\, I_2 \tag{3.33}$$

with complex $\kappa \neq \kappa_0 := q^2(q^2-1)/(q^2+1)$, whereas there are only three independent

$$a_{\kappa_0}{}^{\alpha\beta} = \mathcal{P}_{s}{}^{\alpha\lambda}_{\gamma\delta}(\xi\epsilon x^T)^{\gamma\delta}\epsilon^{\beta\delta}, \tag{3.34}$$

because $a_{\kappa_0}{}^{\alpha\beta}(\epsilon\epsilon^T)_{\beta\alpha} = 0$; the latter belong to the (3,1) irreducible corepresentation of $SU_q(2)\times SU_q'(2)$. There is no other matrix a with the latter property. In the $q=1$ limit (3.34) becomes the familiar

$$a_{\kappa_0}{}^{\alpha\beta} = -\left(\xi\epsilon^{-1}x^T\right)^{(\alpha\lambda)}\epsilon^{\lambda\beta} = -\left\{Im(\xi\,\bar{x})\right\}^{\alpha\beta},$$

where $(\alpha\lambda)$ denotes symmetrization w.r.t. $\alpha\lambda$ and Im the imaginary part.

From (3.15), (3.16), (3.12), (3.14) we easily derive

$$a_\kappa a_\kappa = (1-\kappa)[\xi\bar{\xi}+(1-q^2)\xi\theta\bar{x}]|x|^2 = q^2(1-\kappa)[\xi\bar{\xi}+(1-q^{-2})a_\kappa\theta]|x|^2 \tag{3.35}$$

An analogous statement holds for their primed counterparts. By straightforward calculations one also finds

$$\overline{T}a_\kappa T = -q^{-1}(1+\kappa)[\hat{a}'+\kappa'\theta|x|^2\,I_2] = -q^{-1}(1+\kappa)a'_{\kappa'} \tag{3.36}$$

where $\kappa' := q^2/(1+\kappa)-1$. Looking for a κ such that $\kappa'=\kappa$ we find two solutions $\kappa_\pm = -1\pm q$, which yield the simple changes

$$\bar{T}a_{\kappa_\pm}T = \mp a'_{\kappa_\pm} \tag{3.37}$$

under the 'similarity' transformation T; it is immediate to check that

$$a_{\kappa_+} = -q(dT)\overline{T}|x|^2,$$

which has a well defined limit as $q\to 1$, whereas in the same limit a_{κ_-} diverges. Since $(\bar{\xi}\xi)^{\alpha'\beta'}\in\check{\Omega}^{2\prime}$, which is a $\mathcal{A}$-bimodule, we also find

$$\begin{aligned}
T\bar{\xi}\xi\overline{T} &= x\bar{\xi}\xi\frac{\bar{x}}{|x|^2}q^{-2} \overset{(3.15)}{=} -\xi\bar{x}\xi\frac{\bar{x}}{|x|^2}q^{-2} + (1-q^{-2})\theta|x|^2\xi\frac{\bar{x}}{|x|^2} \\
&\overset{(3.15)}{=} \xi\bar{\xi}\frac{x\bar{x}}{|x|^2}q^{-2} + (q^{-2}-1)\xi\theta\bar{x} + (q^2-1)\theta\xi\bar{x} \\
&= \xi\bar{\xi}q^{-2} + (q^{-2}-q^2)\xi\theta\bar{x} = \xi\bar{\xi}q^2 + (q^2-q^{-2})\hat{a}\theta\ \in\check{\Omega}^{2\prime}
\end{aligned} \tag{3.38}$$

4 Looking for a suitable noncommutative gauge theory framework

We recall some minimal common elements in the formulations of gauge theories on commutative as well as noncommutative spaces [9, 32] (see also [30, 26]). We denote by $\mathcal{A}$ the '$\star$-algebra of functions on the noncommutative space' under consideration, by (d,Ω^*) a differential calculus on $\mathcal{A}$, real in the sense that $d(f^\star) = (df)^\star$. In $U(n)$ gauge theory the

gauge transformations U are unitary $\mathcal{A}$-valued $n \times n$ unitary matrices, $U \in M_n(\mathcal{A}) \equiv M_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{A}$,

$$U^{-1} = U^{\dagger}, \qquad\qquad U \in \mathcal{U}_n. \tag{4.1}$$

Gauge potentials are antihermitean $n \times n$ 1-form-valued matrices $A \equiv (A^{\dot{\alpha}}_{\dot{\beta}})$, $A \in M_n(\Omega^1) \equiv M_n(\mathbb{C}) \otimes_{\mathbb{C}} \Omega^1$. The case $n = 1$ corresponds to electromagnetism. The covariant derivative $D : M_n(\Omega^p) \to M_n(\Omega^{p+1})$ is defined as usual by

$$D\omega_p := d\omega_p + [A, \omega_p\}, \tag{4.2}$$

and is therefore hermitean, $D(f^{\dagger}) = (Df)^{\dagger}$. The associated field strength $F \in M_n(\Omega^2)$ is defined as usual by

$$F := dA + AA. \tag{4.3}$$

At the right-hand side the product AA is both a (row by column) matrix product and a wedge product. It is automatically hermitean. As in commutative geometry, it is immediate to prove that F satisfies the Bianchi identity

$$DF = 0. \tag{4.4}$$

The Yang-Mills equation reads as usual

$$D^{*}F = 0. \tag{4.5}$$

If the exterior derivative can be realized as the graded commutator (3.12) with a special 1-form [9, 53, 32] $-\theta$, then introducing the 1-form-valued matrix $B := -\theta I_n + A$ one finds that

$$F = BB, \qquad\qquad D = [B, \cdot\,] \tag{4.6}$$

and Bianchi identity is now even more trivial. In Connes' noncommutative geometry $-\theta$ is the socalled 'Dirac operator', which has to fulfill more stringent requirements [9].

In commutative geometry the socalled Serre-Swan theorem [49, 10] states that vector bundles over a compact manifold coincide with finitely generated projective modules $\mathcal{E}$ over $\mathcal{A}$. The gauge connection A of a gauge group (fiber bundle) acting on a vector bundle is expressed in terms of the corresponding projector $\mathcal{P}$. Therefore the projectors characterizing the projective modules can be used to completely determine the connections. In Connes' standard approach [9] to noncommutative geometry the finitely generated projective modules are the primary objects to define and develop the gauge theory. The topological properties of the connections can be classified in terms of topological invariants (Chern numbers), and the latter can be computed directly in terms of characters of the projectors (Chern-Connes characters).

Because of the Bianchi identity, in a 4D Riemannian geometry endowed with a (involutive) Hodge map $*$ the YM equation is automatically satisfied by a solution of the (anti)self-duality equations

$$\begin{array}{ll} {}^*F = F & \text{self-duality}, \\ {}^*F = -F & \text{anti-self-duality}. \end{array} \tag{4.7}$$

If Ω^2 splits as in (3.29) then F is uniquely decomposed in a selfdual and an antiselfdual part

$$F = F^+ + F^-. \tag{4.8}$$

Under a gauge transformation U

$$A \to A^U = U^{-1}(AU + dU), \qquad \Leftrightarrow \qquad B \to B^U = U^{-1}BU \tag{4.9}$$

implying as usual

$$F \to F^U = U^{-1}FU. \tag{4.10}$$

The Bianchi identity, the Yang-Mills equation, the (anti)self-duality equations, the splitting (4.8), the flatness condition $F = 0$ are preserved by gauge transformations. As usual, $A = U^{-1}dU$ implies $F = 0$.

Up to normalization factors, the gauge invariant action S and the 'Pontryagin index', or 'second Chern number', Q (a topological invariant) are defined by

$$S = \mathrm{Tr}(F\,{}^*F), \tag{4.11}$$

$$\mathrm{Q} = \mathrm{Tr}(FF), \tag{4.12}$$

where Tr stands for a positive-definite trace (as such, it has to fulfill the cyclic property) combining the $n \times n$-matrix trace with the integral over the noncommutative manifold. If integration $\int$ fulfills itself the cyclic property then this is obtained by simply choosing $\mathrm{Tr} = \int \mathrm{tr}$, where tr stands for the ordinary matrix trace. S is automatically nonnegative.

Q can be computed in terms of the second Chern-Connes character of the projector $\mathcal{P}$ associated to the connection A when Connes' formulation of noncommutative geometry applies.

If, as in the case under discussion, (3.30) holds, S, Q split into the sum, difference of the two nonnegative contributions

$$S = \mathrm{Tr}(F^+\,{}^*F^+) + \mathrm{Tr}(F^-\,{}^*F^-), \tag{4.13}$$

$$\mathrm{Q} = \mathrm{Tr}(F^+\,{}^*F^+) - \mathrm{Tr}(F^-\,{}^*F^-). \tag{4.14}$$

As in the commutative case, these relations imply $S \geq |\mathrm{Q}|$.

In the present $\mathcal{A} \equiv C(\mathbb{R}_q^4) = C(\mathbb{H}_q)$ case the above scheme is not fully applicable because of two **main problems**:

1. Integration over $\mathbb{R}_q^4$ fulfills a *deformed* cyclic property [48].

2. As already recalled, $d(f^\star) \neq (df)^\star$ and there is no $\star$-structure $\star : \Omega^* \to \Omega^*$, but only a $\star$-structure $\star : \mathcal{DC}^* \to \mathcal{DC}^*$ [41], with a rather nonlinear character.

A solution to both problems might be obtained

1. allowing for $\mathcal{DC}^1$-valued A ($\Rightarrow \mathcal{DC}^2$-valued F's), and/or
2. defining a cyclic trace Tr by $\mathrm{Tr}(\omega_4) := \int \mathrm{tr}\left(W^{(1)}\omega_4 W^{(2)}\right)$, with some suitable positive definite $W^{(1)} \otimes W^{(2)} \in M_n(\mathcal{H}) \otimes M_n(\mathcal{H})$ (in Sweedler notation with suppressed sum symbol). (A $W \in M_n(\mathcal{H})$ is a pseudo-differential-operator-valued $n \times n$ matrix).

This hope is based on our results [17]: 1) the Hodge map $*$ is not only $\mathcal{A}$-bilinear, but fully $\mathcal{H}$-bilinear; 2) the $\star$-structure $\star : \mathcal{DC}^* \to \mathcal{DC}^*$ can been recast in a much more suggestive form involving only a similarity transformation with the realization as pseudodifferential operators of the ribbon element $\tilde{w}$ and of the "vector field generators" $\tilde{Z}^i_j$ of the central extension of $U_q so(4)$ with dilatations; 3) d and the exterior coderivative $\delta := - * d *$ become conjugated of each other

$$(\alpha_p, d\beta_{p-1}) = (\delta\alpha_p, \beta_{p-1}), \qquad\qquad (d\beta_{p-1}, \alpha_p) = (\beta_{p-1}, \delta\alpha_p)$$

if one defines

$$(\alpha_p, \beta_p) = \int_{\mathbb{R}^4_q} \alpha_p^\star \;\tilde{w}'^{1/2}\, {}^*\beta_p$$

where $\tilde{w}'$ is the realization of $\tilde{w}$ as a pseudodifferential operator.

5 q-deformed $su(2)$ instanton

We look for $A \in M_2(\Omega^1)$ solutions of the (anti)self-duality equations (4.7) virtually yielding a finite action functional (4.11). Among them we expect deformations of the (multi)instanton solutions of $su(2)$ Yang-Mills theory on the "commutative" $\mathbb{R}^4$. We first recall the instanton solution of Belavin *et al.* [4], which we write down both in t' Hooft [50] and in ADHM [3] quaternion notation:

$$\begin{aligned}
A &= dx^i\,\sigma^a\, \underbrace{\eta^a_{ij} x^j \frac{1}{\rho^2 + r^2/2}}_{A^a_i}, \\
&= -Im\left\{\xi\,\frac{\bar{x}}{|x|^2}\right\}\frac{1}{1+\rho^2\frac{1}{|x|^2}} \\
&= -(dT)\overline{T}\frac{1}{1+\rho^2\frac{1}{|x|^2}} \qquad\qquad (5.1)\\
F &= \xi\bar{\xi}\,\rho^2\frac{1}{(|x|^2+\rho^2)^2}.
\end{aligned}$$

Here $r^2 := x \cdot x = 2|x|^2$, σ^a are the Pauli matrices, η^a_{ij} are the so-called t' Hooft η-symbols, ρ is the size of the instanton (here centered at the origin). The third equality is based on the identity

$$\xi \frac{\bar{x}}{|x|^2} = (dT)\overline{T} + I_2 \frac{d|x|^2}{2|x|^2}$$

and the observation that the first and second term at the rhs are respectively antihermitean and hermitean, i.e. the imaginary and the real part of the quaternion.

In terms of the modified gauge potential $B := A - \theta I_2$ a natural Ansatz for the deformed instanton solution in the 'regular gauge' is (in matrix notation)

$$B = \xi \frac{\bar{x}}{|x|^2}\, l + \theta\, I_2\, m, \tag{5.2}$$

where l, m are functions of x only through $|x|$. For any $f(x)$ we shall denote $f_q(x) := f(qx)$. Using (3.16), (3.14), (3.15), (3.12), (2.10) we find

$$\begin{aligned}
F \;&= B^2 = \xi \frac{\bar{x}}{|x|^2}\, l\, \xi \frac{\bar{x}}{|x|^2}\, l + \xi \frac{\bar{x}}{|x|^2}\, l\, \theta\, m + \theta\, m\, \xi \frac{\bar{x}}{|x|^2}\, l + \theta\, m\, \theta\, m \\
&= \xi \bar{x} \xi \bar{x}\, l_q\, l \frac{q^{-2}}{|x|^4} + \xi \bar{x} \theta\, l_q m \frac{q^{-2}}{|x|^2} + \theta \xi \bar{x}\, m_q l \frac{1}{|x|^2} + \theta^2\, m_q m \\
&= \xi \left[-\bar{\xi} x + (q^2 - 1)\theta |x|^2 \right] \bar{x}\, l_q\, l \frac{q^{-2}}{|x|^4} + \xi [\bar{\xi} + \theta \bar{x}]\, l_q m \frac{q^{-2}}{|x|^2} - \xi \theta \bar{x}\, m_q l \frac{1}{|x|^2} \\
&= \xi \bar{\xi} (m - l)\, l_q \frac{q^{-2}}{|x|^2} + \xi \theta \bar{x} \left[(q^2 - 1) l_q l + l_q m - q^2 m_q l \right] \frac{q^{-2}}{|x|^2}.
\end{aligned}$$

A sufficient condition for F to be selfdual is that the expression in the square bracket vanishes. Setting $h := m/l$ this amounts to the equation $q^2 h_q - h = (q^2 - 1)$, which is solved by

$$m = \left[1 + \bar{\rho}^2 \frac{1}{|x|^2} \right] l,$$

where $\bar{\rho}^2$ is a constant, or might be a further generator of the algebra, commuting with θ. Replacing in the expression for A, F, we find a family of solutions

$$\begin{aligned}
A_l \;&= \xi \bar{x} \tfrac{l}{|x|^2} + \theta I_2 \left\{ 1 + \left[1 + \bar{\rho}^2 \tfrac{1}{|x|^2} \right] l \right\} \\
&= q (dT) \overline{T} l + \theta I_2 \left\{ 1 + \left[q + \bar{\rho}^2 \tfrac{1}{|x|^2} \right] l \right\}, \\
F_l \;&= \xi \bar{\xi} \tfrac{1}{|x|^2} \bar{\rho}^2 \tfrac{q^{-2}}{|x|^2} l_q l,
\end{aligned} \tag{5.3}$$

parametrized by the function l. This large (compared to the undeformed case) freedom in the choice of the solution is due to the fact that we have not yet imposed in A the antihermiticity condition. Actually, we don't

know yet what the 'right' antihermiticity condition is: in fact, for no l is A antihermitean w.r.t. the $\star$-structure [41] mentioned in section 3. In any case, one should check that for the final A the resulting F decreases faster than $|x|^{-2}$ at infinity, so that the resulting action functional (4.11) is finite.

The second term in $(5.3)_1$ is proportional to $d|x|^2$; in the commutative limit $q=1$ it is a connection associated to the noncompact factor $GL^+(1)$ of $\mathbb{H}$. In this limit the antihermiticity condition on A amounts to the vanishing of this term and completely determines the solution. It factors $GL^+(1)$ out of the gauge group to leave a pure $su(2)$ gauge theory. In the q-deformed case we still ignore what the 'right' $\star$- (i.e. Hermitean) structure could be, but it could well happen that w.r.t. the latter the second term in $(5.3)_1$ contains also a antihermitean (i.e. imaginary) part, which would be the connection associated to an additional $U(1)$ factor of the gauge group and which could not be consistently disposed of. In the latter case the associated gauge theory would necessarily be a deformed $u(2)$ one.

For the moment we cannot solve the ambiguity, and content ourselves with writing the solution for a couple of selected choices of l. If we choose l so that the second term in $(5.3)_1$ vanishes and set $\rho^2=\bar{\rho}^2q^{-1}$ we obtain

$$\begin{aligned} A &= -(dT)\overline{T}\frac{1}{1+\rho^2\frac{1}{|x|^2}} \\ F &= q^{-1}\xi\bar{\xi}\frac{1}{q^2|x|^2+\rho^2}\rho^2\frac{1}{|x|^2+\rho^2}. \end{aligned} \tag{5.4}$$

This has manifestly the desired $q\to 1$ limit (5.1). The second choice,

$$l=-\frac{1+q^2}{1+q^4}\frac{1}{1+\tilde{\rho}^2\frac{1}{|x|^2}} \qquad\qquad \tilde{\rho}^2:=\frac{1+q^2}{1+q^4}\bar{\rho}^2,$$

is designed in order that A is proportional to the $a_{\kappa 0}$ of (3.34), so that $A^{\alpha\beta}$ span the (3,1) dimensional, irreducible corepresentation of $SU_q(2)\times SU_q'(2)$. The result is:

$$\begin{aligned} \tilde{A} &= -\frac{1+q^2}{1+q^4}a_{\kappa 0}\frac{1}{|x|^2+\tilde{\rho}^2} \\ \tilde{F} &= \frac{1+q^2}{1+q^4}\xi\bar{\xi}\frac{1}{q^2|x|^2+\tilde{\rho}^2}\tilde{\rho}^2\frac{1}{|x|^2+\tilde{\rho}^2}. \end{aligned} \tag{5.5}$$

This also has the desired $q\to 1$ limit (5.1). If $\bar{\rho}^2\neq 0$, in both cases FF is regular everywhere and decreases as $1/|x|^8$ as $x\to\infty$, therefore it virtually will yield finite action S and Pontryagin index Q upon integration.

As in the undeformed case, to make the determination of multi-instanton solutions easier it is useful to go to the "singular gauge". Note that as in the $q=1$ case $T=x/|x|$ is unitary and formally not continuous at $x=0$, so it can play the role of a 'singular gauge transformation'. In fact A can be obtained through the gauge transformation $A=T(\hat{A}\overline{T}+d\overline{T})$ from the "singular" gauge potential

$$\hat{A} \quad = \quad \overline{T}dT\frac{1}{1+|x|^2\frac{1}{\rho^2}} \tag{5.6}$$

$$
\begin{aligned}
&= -(d\overline{T})T\frac{1}{1+|x|^2\frac{1}{\rho^2}} \\
&\overset{(3.17)}{=} -\left[q^{-1}\bar{\xi}\frac{x}{|x|^2} - \frac{q^{-3}}{q+1}\xi^{\alpha\alpha'}\frac{x^{\beta\beta'}}{|x|^2}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}\right]\frac{1}{1+|x|^2\frac{1}{\rho^2}} \qquad (5.7)\\
\hat{F} &= \overline{T}q^{-1}\xi\bar{\xi}\frac{1}{q^2|x|^2+\rho^2}\rho^2\frac{1}{|x|^2+\rho^2}T, \qquad (5.8)
\end{aligned}
$$

which is the analog of the instanton solution in the "singular gauge" found by 't Hooft in [50]. By singular gauge potential it is meant that it has a pole in $|x|=0$. More generally, the generic solution (5.3) can be obtained through the gauge transformation $A_l = T(\hat{A}_l\overline{T} + d\overline{T})$ from a singular solution $\hat{A}_l$. The latter can be obtained also by starting from an Ansatz like $\hat{B} = \bar{\xi}\frac{x}{|x|^2}\,\hat{l} + \theta\, I_2\,\hat{m}$, instead of (5.2), and imposing that the $\bar{\xi}\xi$ and the $\bar{\xi}\theta x$ term in $\hat{F} = \hat{B}^2$ appear in a combination proportional to (3.38).

A straightforward computation by means of (3.9) shows that $\hat{A}$ can be expressed also in the form

$$\hat{A} = (\hat{\mathcal{D}}\phi)\phi^{-1}, \qquad (5.9)$$

where $\hat{\mathcal{D}}$ is the first-order-differential-operator-valued 2×2 matrix obtained from the expression in the square bracket in (5.7) by the replacement $x^{\alpha\alpha'}/|x|^2 \to q^4\partial^{\alpha\alpha'}$,

$$\hat{\mathcal{D}} := q^3\bar{\xi}\partial - \frac{q}{q+1}dI_2, \qquad (5.10)$$

(for simplicity we are here assuming that ρ^2 commutes with $\xi\partial$) and ϕ is the harmonic potential

$$\phi := 1 + \rho^2\frac{1}{|x|^2}, \qquad \Box\phi| = 0.$$

This is the analog of what happens in the classical case.

The **anti-instanton solution** is obtained just by converting unbarred into barred matrices, and conversely, as in the $q=1$ case. For instance, from (5.4) we obtain the anti-instanton solution in the regular gauge

$$
\begin{aligned}
A' &= -(d\overline{T})T\,\frac{1}{1+\rho^2\frac{1}{|x|^2}}, \\
F' &= q^{-1}\bar{\xi}\xi\frac{1}{|x|^2+\rho^2}\rho^2\frac{1}{q^2|x|^2+\rho^2},
\end{aligned} \qquad (5.11)
$$

and for the one in the singular gauge $\hat{A}' = (\hat{\mathcal{D}}'\phi)\phi^{-1}$, where

$$\hat{\mathcal{D}}' := q^3\xi\bar{\partial} - \frac{q}{q+1}dI_2. \qquad (5.12)$$

Recovering the instanton projective module of [12]

In commutative geometry the instanton projective module $\mathcal{E}$ over $\mathcal{A}$ and the associated gauge connection can be most easily obtained using the quaternion formalism, in the way described e.g. in Ref. [2]. $\mathbb{H} \sim \mathbb{R}^4$ can be compactified as $P^1(\mathbb{H}) \sim S^4$. Let $(w, x) \in \mathbb{H}^2$ be homogenous coordinates of the latter, and choose $w = I_2$ on the chart $\mathbb{H} \sim \mathbb{R}^4$. The element $u \in \mathbb{H}^2$ defined by

$$u \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} I_2 \\ \frac{\rho x}{|x|^2} \end{pmatrix} \left(1 + \frac{\rho^2}{|x|^2}\right)^{-1/2} \tag{5.13}$$

fulfills $u^\dagger u = I_2 \mathbf{1}$, and the 4×2 $\mathcal{A}$-valued matrix u has only three independent components. Therefore the 4×4 $\mathcal{A}$-valued matrix

$$\mathcal{P} := uu^\dagger = \begin{pmatrix} I_2 & \frac{\rho \bar{x}}{|x|^2} \\ \frac{\rho x}{|x|^2} & \frac{\rho^2}{|x|^2} I_2 \end{pmatrix} \frac{1}{1 + \frac{\rho^2}{|x|^2}} \tag{5.14}$$

is a self-adjoint three-dimensional projector. It is the projector associated in the Serre-Swan theorem correspondence to the gauge connection (5.6), by the formula $\hat{A} = u^\dagger du$. The associated projective module $\mathcal{E}$ is embedded in the free module $\mathcal{A}^{16}$ seen as $M_4(\mathcal{A})$, and is obtained from the latter as $\mathcal{E} = \mathcal{P} M_4(\mathcal{A})$.

In the present q-deformed setting we immediately check that the element $u \in \mathbb{H}_q^2$ defined by (5.13) fulfills $u^\dagger u = I_2 \mathbf{1}$ again, so that the 4×2 $\mathcal{A}$-valued matrix $\mathcal{P}$ defined by (5.14) is hermitean and idempotent, and has only 3 independent components. Therefore, it defines the 'instanton projective module' $\mathcal{E} = \mathcal{P} M_4(\mathcal{A})$ also in the q-deformed case. One can easily verify that $\mathcal{P}$ reduces to the hermitean idempotent e of [12] if one chooses the instanton size as $\rho = 1/\sqrt{2}$ and performs the change of generators (2.25). Therefore, interpreting the model [12] as a compactification to S_q^4 of ours, we can use all the results [12] about the Chern-Connes classes of e.

Unfortunately in the q-deformed case it is no more true that $\hat{A} = u^\dagger du$, essentially because the $|x|$-dependent global factor multiplying the matrix at the rhs(5.14) does not commute with the 1-forms of the present calculus ($|x| \xi^i = q \xi^i |x|$).

6 Changing the size and shifting the center of the (anti)instanton

Applying the $\widetilde{SO_q(4)}$ coaction (2.22) to $|x|^2, \xi \bar{x}, \bar{\xi} x$ and using (2.10) we obtain

$$\Delta_L\left(|x|^2\right) = |x|^2 |c|^2,$$

$$\Delta_L(\xi\bar{x}) = |c|^2 a\,\xi\bar{x}\,a^{-1}, \qquad \Delta_L(\bar{\xi}x) = |c|^2\bar{b}\,\bar{\xi}x\,\bar{b}^{-1},$$

$$\Delta_L(\xi\bar{\xi}) = |c|^2 a\,\xi\bar{\xi}\,a^{-1}, \qquad \Delta_L(\bar{\xi}\xi) = |c|^2\bar{b}\,\bar{\xi}\xi\,\bar{b}^{-1},$$

where $|c|^2 := |a|^2|b|^2$. The result is the same also if we consider $|c|^2$ as an independent parameter and choose a, b with unit q-determinant ($|a| = |b| = \mathbf{1}$). If we apply Δ_L to the instanton gauge potentials (5.3) we thus find

$$\begin{aligned}\Delta_L(A_l(\xi,x)) &= aA_l(\xi|c|, x|c|)a^{-1}\\ \Delta_L(F_l(\xi,x)) &= aF_l(\xi|c|, x|c|)a^{-1}.\end{aligned} \tag{6.1}$$

In particular, on the gauge potential (5.4)

$$\begin{aligned}\Delta_L(A) &= -a\,(dT)\overline{T}\frac{1}{1+\rho'^2\frac{1}{|x|^2}}a^{-1}\\ \Delta_L(F) &= a\,\xi\bar{\xi}\frac{1}{q^2|x|^2+\rho'^2}q^{-1}\rho'^2\frac{1}{|x|^2+\rho'^2}a^{-1}\end{aligned} \tag{6.2}$$

where we have set $\rho'^2 := \rho^2|c|^{-2}$. These gauge potentials are again solutions of the self-duality equation, since the latter is covariant under the $\widetilde{SO_q(4)}$ coaction. The result of the $SO_q(4)$ coaction ($|a| = |b| = \mathbf{1}$) can be reabsorbed into a (global) gauge transformation (4.9), with $U = a^{-1}$ (and similarly $U = \bar{b}^{-1}$ for the anti-instanton gauge potentials), i.e. is a gauge equivalent solution. Note that we are thus introducing gauge transformations depending on the additional noncommuting parameters a, b. A full $\widetilde{SO_q(4)}$ coaction ($|c| \neq 1$) instead involves also a change of the size of the instanton, and gives an inequivalent solution. We can thus obtain any size starting from the instanton with unit size.

Having built an (anti)instanton "centered at the origin" with arbitrary size one would like first to translate the center to another point y, then to construct n-instanton solutions "centered at points y_μ", $\mu = 1, 2, \ldots, n$. The appropriate framework is to replace tensor products $\otimes$ by braided tensor products $\underline{\otimes}$ and apply the braided coaddition [34] to the covectors x. This gives new (i.e. gauge inequivalent) solutions. The braided coaddition [34] of the coordinates x reads

$$\underline{\Delta}(x) = x\underline{\otimes}\mathbf{1} + \mathbf{1}\underline{\otimes}x \equiv x - y, \tag{6.3}$$

where we have renamed $x := x\underline{\otimes}\mathbf{1}$, $y := -\mathbf{1}\underline{\otimes}x$. It follows

$$\mathsf{P}_{A\,hk}^{\;ij}y^hy^k = 0 \qquad \Leftrightarrow \qquad y\bar{y} = \bar{y}y = I_2|y|^2 \tag{6.4}$$

Out of the two possible braidings we choose the following one:

$$\begin{aligned} y^hx^i &= q\hat{\mathsf{R}}^{hi}_{\;jk}x^jy^k & &\Leftrightarrow & y^{\alpha\alpha'}x^{\beta\beta'} &= \hat{R}^{\alpha\beta}_{\gamma\delta}\hat{R}^{\alpha'\beta'}_{\gamma'\delta'}x^{\gamma\gamma'}y^{\delta\delta'},\\ \partial_iy^j &= q\hat{\mathsf{R}}^{jh}_{ik}y^k\partial_h & &\Leftrightarrow & \partial_{\alpha\alpha'}y^{\beta\beta'} &= \hat{R}^{\beta\delta}_{\alpha\gamma}\hat{R}^{\beta'\delta'}_{\alpha'\gamma'}y^{\gamma\gamma'}\partial_{\delta\delta'},\\ y^h\xi^i &= q\hat{\mathsf{R}}^{hi}_{\;jk}\xi^jy^k & &\Leftrightarrow & y^{\alpha\alpha'}\xi^{\beta\beta'} &= \hat{R}^{\alpha\beta}_{\gamma\delta}\hat{R}^{\alpha'\beta'}_{\gamma'\delta'}\xi^{\gamma\gamma'}y^{\delta\delta'},\end{aligned} \tag{6.5}$$

(the commutation relations between y and ξ are determined up to a 'conformal factor'; we have fixed the latter in such a way the they look exactly as the commutations relations between x and ξ). As a result,

$$
\begin{aligned}
&d\,y = y\,d, \qquad\qquad y\theta = \theta y, &(6.6)\\
&(x-y)^i\xi^j = q\hat{\mathsf{R}}^{ij}_{\ hk}\xi^h(x-y)^k, &(6.7)\\
&\partial_i(x-y)^j = \delta^i_j + q\hat{\mathsf{R}}^{jh}_{\ ik}(x-y)^k\partial_h, &(6.8)
\end{aligned}
$$

in other words, the differential calculus is invariant under the replacement $x \to x{-}y$ (i.e. under $\underline{\Delta}$). This implies that under this replacement solutions go into solutions. Therefore the instanton solution with "shifted" center y will read in the regular gauge

$$
\begin{aligned}
A &= -d\left[\tfrac{(x-y)}{|x-y|}\right]\tfrac{(\overline{x}-\overline{y})}{|x-y|}\tfrac{1}{1+\rho^2\frac{1}{|x-y|^2}}\\
F &= q^{-1}\xi\bar{\xi}\tfrac{1}{q^2|x-y|^2+\rho^2}\rho^2\tfrac{1}{|x-y|^2+\rho^2}.
\end{aligned}
\qquad (6.9)
$$

and in the singular gauge

$$
\begin{aligned}
\hat{A} &= (\hat{\mathcal{D}}\phi)\phi^{-1}, \qquad\qquad \phi := 1+\rho^2\tfrac{1}{|x-y|^2},\\
\hat{F} &= q^{-1}\tfrac{(\overline{x}-\overline{y})}{|x-y|}\xi\bar{\xi}\tfrac{(x-y)}{|x-y|}\tfrac{1}{q^2|x-y|^2+\rho^2}\rho^2\tfrac{1}{|x-y|^2+\rho^2}.
\end{aligned}
\qquad (6.10)
$$

We conclude this section by sketching how one obtains the 'infinitesimal' version of (6.1), (6.9), i.e. transformations of the solutions under the action of the cross-product $F'{\bowtie}U_q\widetilde{so(4)}$ [40, 33, 19, 22] (i.e. the U.E.A. of the Euclidean quantum group extended with dilatations), where F' is the subalgebra of $\mathcal{H}$ generated by the $\partial_{\alpha\alpha'}$. As known, the (right) action $\triangleleft$ of the dual Hopf algebra H' of a Hopf algebra H can be obtained from the (left) coaction $\Delta_L(v) = v_{(1)}\otimes v_{(2)}$ of the latter (in Sweedler notation) by the rule $v \triangleleft h' = \langle v_{(1)}, h'\rangle\, v_{(2)}$ (here $\langle\ ,\ \rangle$ denotes the pairing between H, H'). For $H' = U_qsu(2)\otimes U'_qsu(2)$ one finds in particular

$$
v^{\alpha\alpha'} \triangleleft gg' = [\tau(g)\,v\,\tau(g')]^{\alpha\alpha'} = \tau^\alpha_\beta(g)v^{\beta\beta'}\tau^{\beta'}_{\alpha'}(g'), \qquad (6.11)
$$

where $v{=}x,\partial,\xi$, $g{\in}U_qsu(2)$, $g'{\in}U'_qsu(2)$ [$gg' = g'g$ in $U_qsu(2){\otimes}U'_qsu(2)$], and τ is the fundamental 2-dim representation of $U_qsu(2)$[§]. One finds the following transformation of the instanton solution A_l under q-rotations:

$$
A_l \triangleleft g' = \varepsilon(g')A_l, \qquad\qquad A_l \triangleleft g = \tau(g_{(1)})A_l\tau(Sg_{(2)}) \qquad (6.12)
$$

where ε, S denotes the counit, antipode of $U_qsu(2), U'_qsu(2)$ and we have used Sweedler notation (with suppressed summation index) for the coproduct $\Delta(g) = g_{(1)}\otimes g_{(2)}$. The transformation law for the antiinstanton solution is obtained exchanging g with g'.

[§]On the FRT [15] generators $L^{\pm\gamma}_{\ \ \delta}$ of $U_qsu(2)$ one has $\tau^\alpha_\beta(L^{\pm\gamma}_{\ \ \delta}) = \hat{R}^{\pm1\gamma\alpha}_{\ \ \ \beta\delta}$.

In section 3 we have introduced partial derivatives ∂_i acting from the left, as conventional. This means that the deformed Leibniz rule takes the form $\partial_{\alpha\alpha'}(ff') = \partial_{\alpha\alpha'}(f)f' + O^{\gamma\gamma'}_{\alpha\alpha'}(f)\partial_{\gamma\gamma'}(f')$, with a suitable linear operator $O^{\gamma\gamma'}_{\alpha\alpha'}$. The generators of infinitesimal translations in the right action $\triangleleft$ are instead derivatives $\overleftarrow{\partial}_{\alpha\alpha'}$ acting from the right, i.e. $ff'\overleftarrow{\partial}_{\alpha\alpha'}= f\,(f'\overleftarrow{\partial}_{\alpha\alpha'})+(f\overleftarrow{\partial}_{\gamma\gamma'})\,\tilde{O}^{\gamma\gamma'}_{\alpha\alpha'}(f')$. The quickest way to determine their action on a function (or differential from) form ω is to recall [35] that this is determined by the equation

$$\omega(x-y) = \omega(x) - (\omega(x)\triangleleft\overleftarrow{\partial}_{\gamma\gamma'})\,y^{\gamma\gamma'} + O(y^2)$$

namely is the coefficient of the term of degree 1 in $-y^{\gamma\gamma'}$ in the expansion of $\omega(x-y)$ in powers of $y^{\gamma\gamma'}$ (put *on the right* of all ξ, x's). One thus easily finds, for instance,

$$A^{\alpha\beta}\triangleleft\overleftarrow{\partial}_{\gamma\gamma'}= q^{-2}\left[A^{\alpha\beta}\frac{x^{\lambda\lambda'}}{\rho^2+|x|^2}+\xi^{\lambda\lambda'}\frac{\delta^{\alpha\beta}}{2(\rho^2+q^2|x|^2)}\right]\epsilon_{\lambda\gamma}\epsilon_{\lambda'\gamma'}-\xi^{\alpha\alpha'}\frac{\epsilon_{\alpha'\gamma'}\epsilon^{\gamma\beta}}{\rho^2+q^2|x|^2} \tag{6.13}$$

on the instanton gauge potentials (5.3). The $\overleftarrow{\partial}_{\alpha\alpha'}$ can be easily realized as elements of $F'\rtimes U_q\widetilde{so(4)}$, or also of the Heisenberg algebra $\mathcal{H}$.

7 Multi-instanton solutions

On the basis of the latter and of the $q=1$ results [50, 51], we first look for n-instanton solutions of the self-duality equation in the "singular gauge" in the form (5.9). Beside the coordinates $x^i \equiv -y_0^i$ we introduce n other coordinates y^i_μ, $\mu = 1, 2, ..., n$ generating as many $\mathbb{R}^4_q$ and braided to each other:

$$\begin{aligned} &\mathsf{P}_A{}^{ij}_{hk}y^h_\mu y^k_\mu = 0 && \Leftrightarrow && y_\mu\bar{y}_\mu = \bar{y}_\mu y_\mu = I_2|y_\mu|^2 \\ &y^h_\nu y^i_\mu = q\hat{\mathsf{R}}^{hi}_{jk}y^j_\mu y^k_\nu && \Leftrightarrow && y^{\alpha\alpha'}_\nu y^{\beta\beta'}_\mu = \hat{R}^{\alpha\beta}_{\gamma\delta}\hat{R}^{\alpha'\beta'}_{\gamma'\delta'}y^{\gamma\gamma'}_\mu y^{\delta\delta'}_\nu \end{aligned} \tag{7.1}$$

with $\mu<\nu$. We shall call $\mathcal{A}_n$ the larger algebra generated by the y^i_μ's and by parameters ρ_μ, $\mu = 1, ..., n$ fulfilling the commutation relations

$$\rho^2_\nu\rho^2_\mu = q^2\,\rho^2_\mu\rho^2_\nu, \qquad \nu<\mu, \tag{7.2}$$

$$\rho^2_\nu y^i_\mu = y^i_\mu\rho^2_\nu\begin{cases} q^{-2} & \nu<\mu, \\ 1, & \nu\geq\mu. \end{cases} \tag{7.3}$$

We shall also enlarge $\mathcal{A}_n$ to the extended Heisenberg algebra $\mathcal{H}_n$ and extended algebra of differential forms $\Omega^*(\mathcal{A}_n)$ by adding as generators the ∂_i and the ξ^i respectively, and to the extended differential calculus algebra $\mathcal{DC}(\mathcal{A}_n)$ by adding as generators both the ξ^i, ∂_i, with cross commutation

relations

$$\rho_\mu^2\xi^i = \xi^i\rho_\mu^2, \qquad\qquad \partial_i\rho_\mu^2 = \rho_\mu^2\partial_i, \tag{7.4}$$

$$\partial_i y_\mu^j = q\hat{\mathsf{R}}_{ik}^{jh} y_\mu^k\partial_h \qquad \Leftrightarrow \qquad \partial_{\alpha\alpha'} y_\mu^{\beta\beta'} = \hat{R}_{\alpha\gamma}^{\beta\delta}\hat{R}_{\alpha'\gamma'}^{\beta'\delta'} y_\mu^{\gamma\gamma'}\partial_{\delta\delta'}, \tag{7.5}$$

$$y_\mu^h\xi^i = q\hat{\mathsf{R}}^{\,hi}_{\;jk}\xi^j y_\mu^k \qquad \Leftrightarrow \qquad y_\mu^{\alpha\alpha'}\xi^{\beta\beta'} = \hat{R}_{\gamma\delta}^{\alpha\beta}\hat{R}_{\gamma'\delta'}^{\alpha'\beta'}\xi^{\gamma\gamma'} y_\mu^{\delta\delta'}, \tag{7.6}$$

Note that the first relations, together with the decomposition $d = \xi^i\partial_i$, imply

$$d\,\rho_\mu^2 = \rho_\mu^2 d. \tag{7.7}$$

Also, from these relations it is evident that $\check{\Omega}^2(\mathcal{A}_n), \check{\Omega}^{2\prime}(\mathcal{A}_n)$ are $\mathcal{A}_n$-bimodules (resp. $\check{\mathcal{DC}}^2(\mathcal{A}_n), \check{\mathcal{DC}}^{2\prime}(\mathcal{A}_n)$ are $\mathcal{H}_n$-bimodules).

In the sequel we shall introduce the short-hand notation

$$z_\mu^i := x^i - v_\mu^i, \qquad v_\mu^i := \sum_{\nu=1}^{\mu} y_\nu^i, \qquad \mu = 1,2,...,n;$$

v_μ^i will play the role of coordinates of the center of the μ-th instanton. It is easy to check from (7.1) that these new n sets of variables generate as many copies of the quantum Euclidean space $\mathbb{R}_q^4$, namely

$$\mathsf{P}_{A\,hk}^{\;ij} z_\mu^h z_\mu^k = 0, \qquad \Leftrightarrow \qquad z_\mu\bar{z}_\mu = \bar{z}_\mu z_\mu = |z_\mu|^2 I_2 \tag{7.8}$$

and together with x^i make up an alternative Poincaré-Birkhoff-Witt basis of the algebra $\mathcal{A}_n$, (i.e. ordered monomials in these variables make up a basis of the vector space underlying $\mathcal{A}_n$). Moreover, differentiating z_μ^j and commuting it with ξ^j is like differentiating and commuting x^j:

$$\partial_i z_\mu^j = \delta_j^i + q\hat{\mathsf{R}}^{\,jh}_{\,ik} z_\mu^k\partial_h, \tag*{(3.4)$_\mu$}$$

$$z_\mu^h\xi^i = q\hat{\mathsf{R}}^{\,hi}_{\;jk}\xi^j z_\mu^k. \tag*{(3.1)$_\mu$}$$

Therefore for any $\mu = 1,2,...,n$ the replacement $x \to z_\mu$ in any true relation involving x, ∂, ξ will generate a new true relation, which we shall label by adding the subscript μ to the original one, as we have just done.

The solution ϕ searched for (5.9) is of the form

$$\phi \equiv \phi_n = 1 + \rho_1^2\frac{1}{|x-y_1|^2} + \rho_2^2\frac{1}{|x-y_1-y_2|^2} + ... + \rho_n^2\frac{1}{|x-y_1-...-y_n|^2} \tag{7.9}$$

or, more compactly,

$$\phi_n = 1 + \sum_{\mu=1}^{n}\rho_\mu^2\frac{1}{|z_\mu|^2},$$

namely a scalar "function" of the coordinates x^i, of the instanton "sizes" ρ_μ and of the "coordinates of their centers". For this to be allowed we have further enlarged $\mathcal{A}_n, \Omega^*(\mathcal{A}_n), \mathcal{H}_n, \mathcal{DC}(\mathcal{A}_n)$ to extended algebras

$\mathcal{A}_n^{ext}, \Omega^*(\mathcal{A}_n^{ext})\mathcal{H}_n^{ext}, \mathcal{DC}(\mathcal{A}_n^{ext})$ by adding as generators inverse elements $1/|z_\mu|$, but we also add the inverses $1/\phi_m$, together with corresponding commutation relations (see the appendix) consistent with the ones given so far.

By Remark 1 and relation (3.10), ϕ is harmonic, exactly as in the classical case. In the appendix we prove more:

Lemma 1 *Denoting* $\phi_q(\{z_i\}) := \phi(\{qz_i\})$,

$$\Box\phi \sim \bar{\partial}\partial\phi = \partial\bar{\partial}\phi = 0 \qquad \textit{(i.e. } \phi \textit{ is harmonic)}, \tag{7.10}$$

$$\phi\xi^i = \xi^i\phi_q, \tag{7.11}$$

$$[\phi, (\partial_i\phi)] = 0 = [\phi, (\partial_h\partial_i\phi)], \tag{7.12}$$

$$\mathsf{P}_{A\,hk}^{\;ij}(\partial^h\phi)(\partial^k\phi) = 0, \tag{7.13}$$

$$(d\phi)(d\phi) = 0. \tag{7.14}$$

We are now ready to prove

Theorem 1 $\hat{A} = (\hat{\mathcal{D}}\phi)\phi^{-1}$ *with* ϕ *defined in (7.9) fulfills the selfduality equation* $(4.7)_1$.

Proof We denote $n_q := 1+q+...+q^{n-1}$. We find

$$d\bar{\xi} = -\bar{\xi}d \stackrel{(3.26)}{=} \frac{-q^2}{1+q^2}\left[\epsilon^{-1}(\xi\bar{\xi}\partial)^T\epsilon+\bar{\xi}\xi\bar{\partial}\right], \tag{7.15}$$

$$\hat{\mathcal{D}}\bar{\xi} \stackrel{(5.10)}{=} q^3\bar{\xi}\partial\bar{\xi} - \frac{q}{q+1}d\bar{\xi} \stackrel{(A.29)}{=} -q\bar{\xi}\xi\bar{\partial} - \frac{q^{-1}3_q}{2_q}d\bar{\xi}, \tag{7.16}$$

$$d\bar{\xi}\partial\phi = -\bar{\xi}d\partial\phi \stackrel{(7.15),(7.10)}{=} \frac{-q^2}{1+q^2}\epsilon^{-1}(\xi\bar{\xi}\partial)^T\epsilon\partial\phi. \tag{7.17}$$

Moreover,

$$\begin{aligned}(\bar{\xi}\partial\phi)(d\phi) &= -d\left[(\bar{\xi}\partial\phi)\phi\right] + (d\bar{\xi}\partial\phi)\phi \\ &\stackrel{(7.11),(7.12)}{=} -d\left[\phi_{q^{-1}}(\bar{\xi}\partial\phi)\right] + (d\bar{\xi}\partial\phi)\phi \\ &= -(d\phi_{q^{-1}})(\bar{\xi}\partial\phi) - \phi_{q^{-1}}(d\bar{\xi}\partial\phi) + (d\bar{\xi}\partial\phi)\phi \\ &\stackrel{(7.11),(7.12)}{=} -(d\bar{\xi}\phi)(\partial\phi) + (d\bar{\xi}\partial\phi)(\phi-\phi_q).\end{aligned} \tag{7.18}$$

Therefore

$$\begin{aligned}\hat{F} &\stackrel{(4.3)}{=} d\left[(\hat{\mathcal{D}}\phi)\phi^{-1}\right] + (\hat{\mathcal{D}}\phi)\phi^{-1}(\hat{\mathcal{D}}\phi)\phi^{-1} \\ &= (d\hat{\mathcal{D}}\phi)\phi^{-1} + (\hat{\mathcal{D}}\phi)\phi^{-1}(d\phi)\,\phi^{-1} + (\hat{\mathcal{D}}\phi)\phi^{-1}(\hat{\mathcal{D}}\phi)\phi^{-1} \\ &\stackrel{(7.11)}{=} (d\hat{\mathcal{D}}\phi)\,\phi^{-1} + (\hat{\mathcal{D}}\phi)\left[(\hat{\mathcal{D}}+d)\phi\right]\phi^{-1}\phi_q^{-1} \\ &\stackrel{(5.10)}{=} (q^3 d\bar{\xi}\partial\phi)\phi^{-1} + (\hat{\mathcal{D}}\phi)\left[\left(q^3\bar{\xi}\partial + \frac{1}{q+1}d\right)\phi\right]\phi^{-1}\phi_q^{-1}\end{aligned}$$

$$
\begin{aligned}
&\stackrel{(7.11)}{=} (q^3 d\bar{\xi}\partial\phi)\phi^{-1} + \left[q^3(\hat{\mathcal{D}}\bar{\xi}\phi_q)\partial\phi + \frac{1}{q+1}(\hat{\mathcal{D}}\phi)d\phi\right]\phi^{-1}\phi_q^{-1}\\
&\stackrel{(7.14),(5.10)}{=} (q^3 d\bar{\xi}\partial\phi)\phi^{-1} + \left\{q\left(\hat{\mathcal{D}}\bar{\xi}\phi\right)(\partial\phi) + \frac{q^3}{q+1}\left(\bar{\xi}\partial\phi\right)(d\phi)\right\}\phi^{-1}\phi_q^{-1}\\
&\stackrel{(7.16),(7.18)}{=} (q^3 d\bar{\xi}\partial\phi)\phi^{-1} - \left\{\left[\left(q^2\bar{\xi}\xi\bar{\partial} + \frac{3_q}{2_q}d\bar{\xi}\right)\phi\right](\partial\phi)\right.\\
&\qquad\qquad\qquad\qquad\left. + \frac{q^3}{2_q}[(d\bar{\xi}\phi)(\partial\phi) - (d\bar{\xi}\partial\phi)(\phi-\phi_q)]\right\}\phi^{-1}\phi_q^{-1}\\
&= q^3(d\bar{\xi}\partial\phi)\left[\phi^{-1} + \frac{1}{2_q}(\phi_q^{-1} - \phi^{-1})\right] - \left[q^2\bar{\xi}\xi(\bar{\partial}\phi)(\partial\phi) + (q^2+1)(d\bar{\xi}\phi)(\partial\phi)\right]\phi_q^{-1}\phi^{-1}\\
&\stackrel{(7.15),(7.17)}{=} \frac{-q^5}{4_q}\left[\epsilon^{-1}(\xi\bar{\xi}\partial)^T\epsilon\partial\phi\right]\left[q\phi^{-1} + \phi_q^{-1}\right] + q^2\epsilon^{-1}(\xi\bar{\xi}\partial\phi)^T\epsilon(\partial\phi)\phi^{-1}\phi_q^{-1};
\end{aligned}
$$

this is a selfdual matrix, $\hat{F} \in M_2\left(\check{\Omega}^2(\mathcal{A}_n)\right)$, because $\xi\bar{\xi}$ is. □

Formally, as $x \to \infty$ also $z_\mu \to \infty$, $\phi \to 1$, and a simple inspection shows that $\hat{A} \to 0$ as $1/|x|^3$, $\hat{F} \to 0$ as $1/|x|^4$, exactly as in the case $q = 1$. Therefore $\hat{F}\hat{F}$ decreases fast enough at infinity for integrals like $\int \mathrm{tr}(\hat{F}\hat{F})$ to be well defined at infinity.

On the other hand, as $z_\mu \to 0$ the function ϕ and therefore the gauge potential $\hat{A}$ are singular, i.e. formally diverge. We don't know yet whether the singularity will cause problems also in a proper functional-analytical treatment (this requires analyzing representations of the algebra). If this is the case then, as in the undeformed theory, the question arises if this singularity is only due to the choice of a singular gauge and can be removed by performing a suitable gauge transformation, or it really affects the field strength. Here we address this issue semi-heuristically. We shall say that an element of our algebra is: 1. analytic in z_μ if its power expansion has no poles in z_μ, i.e. does not depend on $1/|z_\mu|$; 2. regular in z_μ if it formally keeps finite as $z_\mu \to 0$, i.e. in its power expansion the dependence on $1/|z_\mu|$ occurs only through $z_\mu/|z_\mu|$. Since such dependences might change upon changing the order in which the variables $z_1, z_2, ..., z_n$, and possible extra variables $1/|z_1 - z_2|, 1/|z_1 - z_3|, ...$ (if necessary), are displayed, these conditions have to be met for any order. In appendix A.3 we show that performing the "singular gauge transformation" U_2 defined by

$$
U_2 \equiv U_2(z_1, z_2) := \frac{\bar{z}_1}{|z_1|}\frac{y_2}{|y_2|}\frac{\bar{z}_2}{|z_2|} \tag{7.19}
$$

on $\hat{A}_2$ we obtain a 2-istanton solution

$$
A_2 = U_2^{-1}\left(\hat{A}U_2 + dU_2\right) \tag{7.20}
$$

analytic in both z_1, z_2; the corresponding selfdual field strength will be analytic as well. The form of U_2 exactly mimics the undeformed one of Ref. [25, 42]. Of course, for this to make sense, we have to further enlarge

the algebras adding as a generator $1/|y_2|$ with consistent commutation relations; this is done in the subsection A.1. By generalization of the undeformed results [25, 42], we are led to the

Conjecture. Performing the singular gauge transformation U_n recursively defined by $U_0 = \mathbf{1}_2$ and

$$U_n \equiv U_n(z_1, ..., z_n) := U_{n\text{–}1}(z_1, ..., z_{n\text{–}1}) U_{n\text{–}1}^{-1}(y) \frac{\bar{z}_n}{|z_n|}, \tag{7.21}$$

with $U_m(y)$ the function of $y_1, ...y_m$ only defined by $U_m(y) := U_m(z_1 - z_n, ..., z_{n\text{–}1} - z_n)$, we finally obtain a regular n-istanton solution

$$A \equiv A_n = U_n^{-1}\left(\hat{A} U_n + dU_n\right) \tag{7.22}$$

and a corresponding regular selfdual field strength, for any n.

Results for the n**-antiinstanton solutions** are obtained by the already mentioned replacements. In particular, the singular ones $\hat{A}$ are simply obtained replacing $\hat{\mathcal{D}}$ with $\hat{\mathcal{D}}'$ in (5.9).

A Appendix

A.1 Additional relations for the extended algebra

Let $z := x\text{–}y$, where y is defined as in section 6. Let $a \cdot b := a^{\alpha\alpha'} b^{\beta\beta'} \epsilon_{\alpha\beta} \epsilon_{\alpha'\beta'}$. The following relations are consequences of the commutation relations for the generators x^i, y^i, z^i, ρ_x or are (the only) consistent extensions of these consequences to the square root, inverse, and inverse square root of $|z|^2, |x|^2, |y|^2$ having the desired, commutative $q \to 1$ limit.

$$C\xi^i = q^2 \xi^i C, \qquad \text{for} \qquad C = |x|^2, |y|^2, y \cdot x\,,\; x \cdot y, |z|^2 \tag{A.1}$$

$$\gamma \xi^i = q^{-1} \xi^i \gamma, \qquad \text{for} \qquad \gamma = \frac{1}{|x|}, \frac{1}{|y|}, \frac{1}{|z|} \tag{A.2}$$

$$y^i |x|^2 = q^2 |x|^2 y^i, \qquad x^i |y|^2 = q^{-2} |y|^2 x^i, \qquad x \cdot y = q^2 y \cdot x, \tag{A.3}$$

$$y^i |x|^{\pm 1} = q^{\mp 1} |x|^{\pm 1} y^i, \qquad x^i |y|^{\pm 1} = q^{\pm 1} |y|^{\pm 1} x^i, \qquad \frac{1}{|y|}\frac{1}{|x|} = \frac{q}{|x|}\frac{1}{|y|}, \tag{A.4}$$

$$[y^i, y \cdot x] = |y|^2 x^i (1 - q^{-2}), \qquad [x^i, y \cdot x] = -y^i |x|^2 (1 - q^{-2}), \tag{A.5}$$

$$z^i \frac{x^j}{|x|^2} = q^{-1} \hat{\mathsf{R}}^{ij}_{hk} \frac{x^h}{|x|^2} z^k + (1 - q^{-2}) g^{ij}, \tag{A.6}$$

$$z \frac{\bar{x}}{|x|^2} = -q^{-2} \frac{x}{|x|^2} \bar{z} + q^{-4} \frac{x}{|x|^2} \cdot z + (1 - q^{-4}), \tag{A.7}$$

$$\frac{x^i}{|x|^2}|z|^2=q^2|z|^2\frac{x^i}{|x|^2}+(1-q^2)z^i, \tag{A.8}$$

$$\frac{1}{|z|^2}\frac{x^i}{|x|^2}=\frac{x^i}{|x|^2}\frac{q^2}{|z|^2}+(1-q^2)\frac{z^i}{|z|^4}, \tag{A.9}$$

$$\frac{x^i}{|x|^2}|z| = q|z|\frac{x^i}{|x|^2}+(1-q)\frac{z^i}{|z|}, \tag{A.10}$$

$$\frac{1}{|z|}\frac{x^i}{|x|^2} = \frac{x^i}{|x|^2}\frac{q}{|z|}+(1-q)\frac{z^i}{|z|^3}, \qquad \frac{q}{|z|}\frac{y^i}{|y|^2} = \frac{y^i}{|y|^2}\frac{1}{|z|}+(1-q)\frac{z^i}{|z|^3} \tag{A.11}$$

$$|z|^2|x|^2=|x|^2\left[q^4|z|^2+(1-q^2)x\cdot z\right], \tag{A.12}$$

$$|z|^2\frac{1}{|x|^2}=\frac{q^{-4}}{|x|^2}|z|^2+(1-q^{-2})\left[q^{-4}\frac{x}{|x|^2}\cdot z+(1-q^{-4})\right] \tag{A.13}$$

$$\frac{1}{|z|}\frac{1}{|x|^2}=\frac{q^2}{|x|^2}\frac{1}{|z|}+(q^{-1}-1)\frac{x}{|x|^2}\cdot\frac{z}{|z|^3} \tag{A.14}$$

$$\left[\xi^h\frac{x^i}{|x|^2},|z|^2\right] = (1-q^2)\xi^h z^i, \tag{A.15}$$

$$\left[\xi^h\frac{x^i}{|x|^2},\frac{1}{|z|}\right] = (1-q^{-1})\xi^h\frac{z^i}{|z|^3}, \tag{A.16}$$

$$\left[Td\overline{T},\frac{1}{|z|}\right]=(1-q^{-1})T_z d\overline{T_z}\frac{1}{|z|}. \tag{A.17}$$

$$\frac{|x|}{|\rho_x|}\ \text{ commutes with } x^i,y^i,z^i,|x|,|y|,|z| \tag{A.18}$$

For instance, relations (A.2) are postulated by consistency with (A.1). Relation (A.7) follows from (A.4), (2.21), (2.19), (2.14), $(6.5)_1$. Eq. (A.8), (A.9) follow from the preceding ones. Relations (A.10), (A.11) are postulated by consistency with (A.7), (A.8). Eq. (A.15) follows from (A.8),(A.9) and (A.1). Relation (A.16) follows from (A.10), (A.11) and (A.2). Eq. (A.17), where we have set $T_z:=z/|z|$, is a particular consequence of (A.16). Eq.(A.18) follows from (7.3).

From (A.4) it also follows

$$\frac{1}{|x|}z = z\frac{q}{|x|} + (1-q)\frac{x}{|x|}, \qquad \frac{1}{|y|}z = z\frac{q^{-1}}{|y|} + (q^{-1}-1)\frac{y}{|y|} \tag{A.19}$$

$$\begin{aligned} &|x|\frac{1}{|y|}x = q^{-1}x|x|\frac{1}{|y|}, \qquad |x|\frac{1}{|y|}y = q^{-1}y|x|\frac{1}{|y|}, \\ &|x|\frac{1}{|y|}z=z|x|\frac{q^{-1}}{|y|}, \qquad |x|\frac{1}{|y|}\frac{1}{|z|}=\frac{q}{|z|}|x|\frac{1}{|y|}, \qquad \left[|x|\frac{1}{|y|},\frac{z}{|z|}\right]=0. \end{aligned} \tag{A.20}$$

Lemma 2

$$z\frac{\bar{y}}{|y|}\frac{x}{|x|}=\frac{x}{|x|}\frac{\bar{y}}{|y|}z, \qquad \bar{z}\frac{y}{|y|}\frac{\bar{x}}{|x|}=\frac{\bar{x}}{|x|}\frac{y}{|y|}\bar{z}, \tag{A.21}$$

$$\bar{z}\frac{x}{|x|}\frac{\bar{y}}{|y|}=\frac{\bar{y}}{|y|}\frac{x}{|x|}\bar{z}, \qquad z\frac{\bar{x}}{|x|}\frac{y}{|y|}=\frac{y}{|y|}\frac{\bar{x}}{|x|}z. \tag{A.22}$$

Proof We use (A.20) and (A.8)

$$\bar{z}\frac{y}{|y|}\frac{\bar{x}}{|x|}\stackrel{(A.4)}{=}\bar{z}y\frac{\bar{x}}{|x|}\frac{1}{|y|}=\bar{z}(x-z)\frac{\bar{x}}{|x|}\frac{1}{|y|}\stackrel{(2.10)}{=}\left[\bar{z}|x|-|z|^2\frac{\bar{x}}{|x|}\right]\frac{1}{|y|}, \tag{A.23}$$

whereas

$$\begin{aligned}\frac{\bar{x}}{|x|}\frac{y}{|y|}\bar{z}&=\frac{\bar{x}}{|x|}\frac{1}{|y|}(x-z)\bar{z}\stackrel{(A.4)}{=}\frac{\bar{x}}{|x|}\left(x\frac{q^{-1}}{|y|}-\frac{1}{|y|}z\right)\bar{z}\\&\stackrel{(2.10),(7.8)}{=}\left(|x|\frac{q^{-1}}{|y|}\bar{z}-\frac{\bar{x}}{|x|}\frac{1}{|y|}|z|^2\right)\stackrel{(A.20)}{=}q^{-2}\left(\bar{z}-\frac{\bar{x}}{|x|^2}|z|^2\right)|x|\frac{1}{|y|}\\&\stackrel{(A.8)}{=}\left(\bar{z}|x|-|z|^2\frac{\bar{x}}{|x|}\right)\frac{1}{|y|}=rhs(A.23).\end{aligned}$$

Completely analogous is the proof of the other relations. □

Proposition 2

$$\left[|z|^2,\frac{\bar{y}}{|y|}\frac{x}{|x|}\right]=\left[|z|^2,\frac{y}{|y|}\frac{\bar{x}}{|x|}\right]=\left[|z|^2,\frac{x}{|x|}\frac{\bar{y}}{|y|}\right]=\left[|z|^2,\frac{\bar{x}}{|x|}\frac{y}{|y|}\right]=0, \tag{A.24}$$

$$\left[|z|^{\pm1},\frac{\bar{y}}{|y|}\frac{x}{|x|}\right]=\left[|z|^{\pm1},\frac{y}{|y|}\frac{\bar{x}}{|x|}\right]=\left[|z|^{\pm1},\frac{x}{|x|}\frac{\bar{y}}{|y|}\right]=\left[|z|^{\pm1},\frac{\bar{x}}{|x|}\frac{y}{|y|}\right]=0, \tag{A.25}$$

$$U_2(x,z):=\frac{\bar{z}}{|z|}\frac{y}{|y|}\frac{\bar{x}}{|x|}=\frac{\bar{x}}{|x|}\frac{y}{|y|}\frac{\bar{z}}{|z|}=\left(\frac{\bar{z}}{|z|^2}-\frac{\bar{x}}{|x|^2}\right)|x|\frac{1}{|y|}|z|, \tag{A.26}$$

$$U_2^{-1}(x,z)=\frac{z}{|z|}\frac{\bar{y}}{|y|}\frac{x}{|x|}=\frac{x}{|x|}\frac{\bar{y}}{|y|}\frac{z}{|z|}=\left(\frac{z}{|z|^2}-\frac{x}{|x|^2}\right)|x|\frac{1}{|y|}|z|, \tag{A.27}$$

$$\left[|z|^{\pm1},U_2(x,z)\right]=0, \tag{A.28}$$

Proof Eq. (A.24) are direct consequences of $|z|^2=\bar{z}z=z\bar{z}$ and of the relations in the lemma. (A.25) are derived by consistency with (A.24). The first equality in (A.26) is a direct consequence of $(A.21)_2$ and of (A.25); the second equality is a consequence of (A.23), (A.10), (A.20). Eq. (A.28) follows from (A.25) and $[z,|z|]=[\bar{z},|z|]=0$. □

Relations (3.5), (7.6), (3.1), $(6.5)_2$ respectively imply

$$\xi\bar{\partial}+q^2\partial\bar{\xi}=q^{-2}dI_2, \qquad \bar{\xi}\partial+q^2\bar{\partial}\xi=q^{-2}dI_2, \tag{A.29}$$

$$\xi\bar{y}+y\bar{\xi}=q^{-2}(\xi\cdot y)I_2, \qquad \bar{\xi}y+\bar{y}\xi=q^{-2}\,(\xi\cdot y)\,I_2, \tag{A.30}$$

$$\xi\bar{z}+z\bar{\xi}=q^{-2}(\xi\cdot z)I_2, \qquad \bar{\xi}z+\bar{z}\xi=q^{-2}\,(\xi\cdot z)\,I_2, \tag{A.31}$$

$$x\bar{y}+y\bar{x}=q^{-2}(x\cdot y)I_2, \qquad \bar{x}y+\bar{y}x=q^{-2}\,(x\cdot y)\,I_2, \tag{A.32}$$

A quick way to prove these relations is to note that they can be obtained from (3.15) by the following replacements: $x/|x|^2q^2(1-q^2)\to\partial$ (see Remark 1), $x\to y$, $x\to z$, $x\to y$ and $\xi\to x$, respectively.

Remark 2. By (A.19), reordering $1/|x|,1/|y|$ w.r.t. x^i,z^i does not introduce additional powers of $1/|x|,1/|y|,1/|z|$. Consequently, for any $f(x,z)$ analytic w.r.t. x,z, $f\,1/|x|=(1/|x|)g$, $f\,1/|y|=(1/|y|)h$, with $g(x,z)$, $h(x,z)$ analytic functions w.r.t. x,z. By (A.14), reordering $1/|z|$ w.r.t. $1/|x|^2$ does not introduce additional powers of $1/|x|$.

Remark 3. Any relation (...) or Remark proved/postulated so far in this appendix is mapped into a new true/consistent one, which we shall label as $(...)_\mu$ or $(...)_{\mu\nu}$ according to the cases, by the replacements $x\to z_\mu$, $\rho_x\to\rho_\mu$, $y\to\sum_{\lambda=\mu+1}^{\nu}y_\lambda$, $z\to z_\nu$, $\rho_z\to\rho_\nu$ with $\nu>\mu$.

A.2 Proof of Lemma 1

Relation (7.10) is a straightforward consequence of $(7.4)_2$ and of $(3.10)_\mu$, $\mu=1,2,...,n$. Relation (7.11) is a straightforward consequence of $(7.4)_1$ and of $(A.2)_\mu$. To prove (7.12), (7.13) we first state the following relations:

$$\frac{1}{|z_\nu|^2}y^i_\mu=y^i_\mu\frac{q^2}{|z_\nu|^2}, \qquad \frac{\rho^2_\nu}{|z_\nu|^2}y^i_\mu=y^i_\mu\frac{\rho^2_\nu}{|z_\nu|^2}, \qquad [\frac{\rho^2_\nu}{|z_\nu|^2},z^i_\nu]=0 \qquad \nu<\mu, \tag{A.33}$$

$$\left[\frac{\rho^2_\nu}{|z_\nu|^2},z^k_\mu\right]=0, \qquad \left[\frac{\rho^2_\nu}{|z_\nu|^2},\frac{z^k_\mu}{|z_\mu|}\right]=0, \qquad \left[\frac{\rho^2_\nu}{|z_\nu|^2},\rho^2_\mu\right]=0, \qquad \nu\le\mu \tag{A.34}$$

$$\left[\frac{\rho^2_\nu}{|z_\nu|^2},\frac{\rho^2_\mu}{|z_\mu|^2}\right]=0, \tag{A.35}$$

$$\left[\frac{\rho^2_\nu}{|z_\nu|^2},\frac{z^k_\mu}{|z_\mu|^4}\rho^2_\mu\right]=(1-q^2)\frac{z^k_\nu}{|z_\nu|^4}\rho^2_\nu\frac{\rho^2_\mu}{|z_\mu|^2} \qquad \nu>\mu \tag{A.36}$$

$$\mathsf{P_A}{}^{ij}_{hk}z^h_\mu z^k_\nu=-\mathsf{P_A}{}^{ij}_{hk}z^h_\nu z^k_\mu \tag{A.37}$$

$$|z_\nu|^2\mathsf{P_A}{}^{ij}_{hk}z^h_\mu z^k_\nu\frac{1}{|z_\mu|^2}=-q^2\mathsf{P_A}{}^{ij}_{hk}z^h_\nu z^k_\mu\frac{1}{|z_\mu|^2}|z_\nu|^2 \qquad \nu>\mu \tag{A.38}$$

Relations (A.33) follow from $(A.4)_{\mu\nu}$ and (7.3). The first two relations (A.34) follow from (A.33), the third from (7.3). (A.35) is an immediate

consequence of (A.34). Relation (A.36) is a consequence of (A.34) and $(A.9)_{\mu\nu}$, (A.37) a consequence of (7.8), (7.1) and (2.14), (A.38) a consequence of (A.37), $(7.8)_\mu$ $(A.8)_\mu$.

To prove $(7.12)_1$ we proceed as follows:

$$
\begin{aligned}
\phi(\partial^k\phi) &= -q^{-4}\left[1+\sum_{\mu=0}^{n}\frac{\rho_\mu^2}{|z_\mu|^2}\right]\sum_{\nu=0}^{n}\frac{z_\nu^k}{|z_\nu|^4}\rho_\nu^2 \\
&= (\partial^k\phi) - q^{-4}\left[\sum_{\mu=0}^{n}\frac{\rho_\mu^2}{|z_\mu|^2}\frac{z_\mu^k}{|z_\mu|^4}\rho_\mu^2+\sum_{\substack{\mu,\nu=0\\ \nu>\mu}}^{n}\frac{\rho_\mu^2}{|z_\mu|^2}\frac{z_\nu^k}{|z_\nu|^4}\rho_\nu^2+\sum_{\substack{\mu,\nu=0\\ \nu<\mu}}^{n}\frac{\rho_\mu^2}{|z_\mu|^2}\frac{z_\nu^k}{|z_\nu|^4}\rho_\nu^2\right] \\
&\overset{(A.36)}{=} (\partial^k\phi) - q^{-4}\left[\sum_{\mu=0}^{n}\frac{z_\mu^k}{|z_\mu|^4}\rho_\mu^2\frac{\rho_\mu^2}{|z_\mu|^2}+q^2\sum_{\substack{\mu,\nu=0\\ \nu>\mu}}^{n}\frac{z_\nu^k}{|z_\nu|^4}\rho_\nu^2\frac{\rho_\mu^2}{|z_\mu|^2}\right. \\
&\left.+\sum_{\substack{\mu,\nu=0\\ \nu<\mu}}^{n}\left(\frac{z_\nu^k}{|z_\nu|^4}\rho_\nu^2\frac{1}{|z_\mu|^2}\rho_\mu^2 - qk\frac{z_\mu^k}{|z_\mu|^4}\rho_\mu^2\frac{\rho_\nu^2}{|z_\nu|^2}\right)\right] \\
&= (\partial^k\phi) - q^{-4}\sum_{\nu=0}^{n}\frac{z_\nu^k}{|z_\nu|^4}\rho_\nu^2\left[\sum_{\mu=0}^{n}\frac{\rho_\mu^2}{|z_\mu|^2}\right] = (\partial^k\phi)\,\phi
\end{aligned}
$$

(in the fourth equality we have used the fact that the second term in the inner bracket is proportional to and therefore can be put together with the second in the square bracket). Similar is the proof of $(7.12)_2$. As a consequence we have also $[\phi^{-1},(\partial^k\phi)]=0$ and, by the replacement $z_\mu \to qz_\mu$, $[{\phi_q}^{-1},(\partial^k\phi_q)]=0$. We now prove (7.13)

$$
\begin{aligned}
&\mathsf{P}_{A\,hk}^{\;ij}(\partial^h\phi)(\partial^k\phi) \\
&\sim \mathsf{P}_{A\,hk}^{\;ij}\left[\sum_{\mu=0}^{n}\frac{z_\mu^h}{|z_\mu|^4}\rho_\mu^2\frac{z_\mu^k}{|z_\mu|^4}\rho_\mu^2+\sum_{\substack{\mu,\nu=0\\ \nu>\mu}}^{n}\frac{z_\mu^h}{|z_\mu|^4}\rho_\mu^2\frac{z_\nu^k}{|z_\nu|^4}\rho_\nu^2+\sum_{\substack{\mu,\nu=0\\ \nu<\mu}}^{n}\frac{z_\mu^h}{|z_\mu|^4}\rho_\mu^2\frac{z_\nu^k}{|z_\nu|^4}\rho_\nu^2\right] \\
&= \mathsf{P}_{A\,hk}^{\;ij}\left[\sum_{\mu=0}^{n}\frac{z_\mu^h z_\mu^k}{|z_\mu|^8}\rho_\mu^4+\sum_{\substack{\mu,\nu=0\\ \nu>\mu}}^{n}\frac{z_\mu^h}{|z_\mu|^4}\rho_\mu^2\frac{z_\nu^k}{|z_\nu|^4}\rho_\nu^2+\sum_{\substack{\mu,\nu=0\\ \mu<\nu}}^{n}\frac{z_\nu^h}{|z_\nu|^4}\rho_\nu^2\frac{z_\mu^k}{|z_\mu|^4}\rho_\mu^2\right]
\end{aligned}
$$

The first term in the square bracket vanishes because of (7.8), whereas, because of (A.34-A.36), the other two give, as claimed,

$$
\begin{aligned}
&\mathsf{P}_{A\,hk}^{\;ij}(\partial^h\phi)(\partial^k\phi) \sim \mathsf{P}_{A\,hk}^{\;ij}\sum_{\substack{\mu,\nu=0\\ \nu>\mu}}^{n}\left[\frac{1}{|z_\mu|^2}z_\mu^h z_\nu^k\frac{1}{|z_\nu|^2}\frac{\rho_\mu^2}{|z_\mu|^2}\frac{\rho_\nu^2}{|z_\nu|^2}+\right. \\
&\left.\frac{z_\nu^h}{|z_\nu|^2}\left(\frac{z_\mu^k}{|z_\mu|^4}\rho_\mu^2\frac{\rho_\nu^2}{|z_\nu|^2}+(1-q^2)\frac{z_\nu^h}{|z_\nu|^4}\rho_\nu^2\frac{\rho_\mu^2}{|z_\mu|^2}\right)\right]
\end{aligned}
$$

$$\stackrel{(A.37),(7.8)_\mu}{=} \mathsf{P}_{A}{}^{ij}_{hk} \sum_{\substack{\mu,\nu=0\\ \nu>\mu}}^{n} \left[-\frac{1}{|z_\mu|^2} z_\nu^h z_\mu^k \frac{1}{|z_\nu|^2} \frac{\rho_\mu^2}{|z_\mu|^2} \frac{\rho_\nu^2}{|z_\nu|^2} + \frac{z_\nu^h}{|z_\nu|^2} \frac{z_\mu^k}{|z_\mu|^4} \rho_\mu^2 \frac{\rho_\nu^2}{|z_\nu|^2}\right]$$

$$= \mathsf{P}_{A}{}^{ij}_{hk} \sum_{\substack{\mu,\nu=0\\ \nu>\mu}}^{n} \left[-q^2 z_\nu^h \frac{z_\mu^k}{|z_\mu|^2} \frac{1}{|z_\nu|^2} + q^2 z_\nu^h \frac{z_\mu^k}{|z_\mu|^2} \frac{1}{|z_\nu|^2}\right] \frac{\rho_\mu^2}{|z_\mu|^2} \frac{\rho_\nu^2}{|z_\nu|^2} = 0;$$

in the last equality we have used (A.38), $(7.8)_\mu$, $(A.8)_\mu$. Finally,

$$\begin{aligned}
&(d\phi)(d\phi) = (d\phi)(\xi^i\partial_i\phi) \stackrel{(7.11)}{=} (d\xi^i\phi_q)(\partial_i\phi) = -q^{-2}\xi^i(d\phi)(\partial_i\phi)\\
&= -q^{-2}\xi^i\xi^j(\partial_j\phi)(\partial_i\phi) \stackrel{(3.2)}{=} -q^{-2}\xi^h\xi^k \mathsf{P}_{A}{}^{ij}_{hk} g_{jl} g_{im} (\partial^l\phi)(\partial^m\phi)\\
&= -q^{-2}\xi^h\xi^k g_{hj} g_{ki} \mathsf{P}_{A}{}^{ij}_{lm} (\partial^l\phi)(\partial^m\phi) \stackrel{(7.13)}{=} 0
\end{aligned}$$

proves (7.14). In the last but one equality we have used the property (see e.g. [15, 19]) $[\mathsf{P}_A, P(g\otimes_{\mathbb{C}} g)] = 0$, where P denotes the permutation matrix.

A.3 Proof of the analyticity of A_2

For any quaternion w let $V(w) := w/|w|$. As a consequence, $V^{-1}(w) := \bar{w}/|w|$. So $T = V(x)$. We shall use also the shorter notation $T_n := V(z_n)$. Having defined U_2 as in (7.19), we find

$$U_2^{-1}(dU_2) = T_2 V^{-1}(y_2) T_1 (d\overline{T}_1) V(y_2) \bar{T}_2 + T_2 (d\overline{T}_2). \tag{A.39}$$

From the definition (7.9) it follows, for both $\mu = 1, 2$,

$$\phi_2{}^{-1} = \frac{|z_\mu|^2}{\rho_\mu^2} f_\mu, \text{where } f_\mu \text{ is analytic in } z_\mu. \tag{A.40}$$

Using properties (A.35), (A.34) we immediately find for $m = 1, 2$

$$[\phi_m, T_2] = 0, \qquad\qquad [\phi_1, \phi_2] = 0, \tag{A.41}$$

whereas we find, as consequences of (A.18), (A.25), (7.3)

$$\left[\phi_m, T_1 V^{-1}(y_2)\right] = \left[\phi_m, V(y_2)\overline{T}_1\right] = [\phi_m, U_2] = 0. \tag{A.42}$$

Moreover, by straightforward calculations,

$$(\hat{\mathcal{D}}\phi_2) = -(d\overline{T}_1)T_1 \frac{\rho_1^2}{|z_1|^2} - (d\overline{T}_2)T_2 \frac{\rho_2^2}{|z_2|^2}. \tag{A.43}$$

We first show that A_2 is an analytic function of z_1:

$$\begin{aligned}
A_2 &\stackrel{(7.20),(5.9)}{=} U_2^{-1}\left[(\hat{\mathcal{D}}\phi_2)\phi_2^{-1}U_2 + dU_2\right]\\
&\stackrel{(A.43)}{=} U_2^{-1}\left[-\sum_{\mu=1}^{2}(d\overline{T}_\mu)T_\mu \frac{\rho_\mu^2}{|z_\mu|^2}\phi_2^{-1}U_2 + dU_2\right]
\end{aligned}$$

$$\stackrel{(A.42)}{=} U_2^{-1}\left[-\sum_{\mu=1}^{2}(d\overline{T}_\mu)T_\mu U_2\,\frac{\rho_\mu^2}{|z_\mu|^2}\phi_2^{-1}+dU_2\right]$$

$$\stackrel{(A.39)}{=} U_2^{-1}\left[-(d\overline{T}_1)T_1\overline{T}_1V(y_2)\overline{T}_2\frac{\rho_1^2}{|z_1|^2}\phi_2^{-1}+(d\overline{T}_1)V(y_2)\overline{T}_2\right.$$
$$\left.-(d\overline{T}_2)T_2\,U_2\phi_2^{-1}\frac{\rho_2^2}{|z_2|^2}\right]+T_2(d\overline{T}_2)$$

$$=U_2^{-1}(d\overline{T}_1)V(y_2)\overline{T}_2\left(\phi_2-\frac{\rho_1^2}{|z_1|^2}\right)\phi_2^{-1}$$
$$-U_2^{-1}(d\overline{T}_2)T_2\,U_2\phi_2^{-1}\frac{\rho_2^2}{|z_2|^2}+T_2(d\overline{T}_2)$$

$$\stackrel{(A.40)}{=} T_2V^{-1}(y_2)T_1(d\overline{T}_1)V(y_2)\overline{T}_2\left(1+\frac{\rho_2^2}{|z_2|^2}\right)\frac{|z_1|^2}{\rho_1^2}f_1$$
$$-T_1V^{-1}(y_2)T_2(d\overline{T}_2)\,V(y_2)\overline{T}_1\frac{|z_1|^2}{\rho_1^2}f_1\frac{\rho_2^2}{|z_2|^2}+T_2(d\overline{T}_2)$$

$$\stackrel{(A.18)_{12},(A.2)_1}{=} T_2V^{-1}(y_2)T_1(d\overline{T}_1)\frac{|z_1|^2}{\rho_1^2}V(y_2)\overline{T}_2\left(1+\frac{\rho_2^2}{|z_2|^2}\right)f_1$$
$$-q^{-1}\frac{z_1}{\rho_1}V^{-1}(y_2)T_2(d\overline{T}_2)\,V(y_2)\frac{\bar{z}_1}{\rho_1}f_1\frac{\rho_2^2}{|z_2|^2}+T_2(d\overline{T}_2).$$

Looking at (3.17) we see that $T_1(d\overline{T}_1)|z_1|^2/\rho_1^2$ is analytic in z_1; the factors at its left and right also are. The second term is also a product of analytic factors in z_1. Therefore, by Remark 2, the first two terms at the rhs are analytic in z_1, however we fix the order of the variables $z_1, z_2, 1/|y_2|$. Finally, the term $T_2(d\overline{T}_2)$ is independent of z_1. We conclude that A_2 is analytic in z_1.

We now show that A_2 is analytic in z_2. We first prove that $\hat{A}_2^{\overline{T}_2} := T_2[\hat{A}_2\overline{T}_2+(d\overline{T}_2)]$ is a regular function of z_2, more precisely, even analytic.

$$\hat{A}_2^{\overline{T}_2} := T_2[\hat{A}_2\overline{T}_2+d\overline{T}_2] \stackrel{(5.9)}{=} T_2\left[(\hat{\mathcal{D}}\phi_2)\phi_2^{-1}\overline{T}_2+(d\overline{T}_2)\right]$$

$$\stackrel{(A.42),(A.43)}{=} T_2\left[-(d\overline{T}_2)T_2\overline{T}_2\phi_2^{-1}\frac{\rho_2^2}{|z_2|^2}+(d\overline{T}_2)-(d\overline{T}_1)T_1\overline{T}_2\phi_2^{-1}\frac{\rho_1^2}{|z_1|^2}\right]$$

$$\stackrel{(2.12)}{=} T_2(d\overline{T}_2)\phi_2^{-1}\left(\phi_2-\frac{\rho_2^2}{|z_2|^2}\right)-\frac{z_2}{|z_2|}(d\overline{T}_1)T_1\overline{T}_2\phi_2^{-1}\frac{\rho_1^2}{|z_1|^2}$$

$$\stackrel{(A.17)_{12}}{=} T_2(d\overline{T}_2)\phi_2^{-1}\phi_1-z_2\left[(d\overline{T}_1)T_1-(1-q^{-1})(d\overline{T}_2)T_2\right]\frac{\overline{T}_2}{|z_2|}\phi_2^{-1}\frac{\rho_1^2}{|z_1|^2}$$

$$\stackrel{(2.12),(A.2)_2}{=} T_2(d\overline{T}_2)\phi_2^{-1}\left[\phi_1+(q-1)(\phi_1-1)\right]-z_2(d\overline{T}_1)T_1\frac{\rho_1^2}{|z_1|^2}\frac{\bar{z}_2}{|z_2|^2}\phi_2^{-1}$$

$$\stackrel{(A.40)}{=} T_2(d\overline{T}_2)\frac{|z_2|^2}{\rho_2^2}f_2\left[q\phi_1+(1-q)\right]-z_2(d\overline{T}_1)T_1\frac{\rho_1^2}{|z_1|^2}\frac{\bar{z}_2}{\rho_2^2}f_2$$

As ϕ_1 does not depend on z_2 and $T_2(d\overline{T}_2)|z_2|^2, f_2$ are analytic in z_2, the

first term is. On the other hand, the second term is manifestly analytic in z_2. Now, by a further gauge transformation $\tilde{U} := V(y_2)\overline{T}_1$,

$$A_2 = \tilde{U}^{-1}\hat{A}^{\overline{T}_2}\tilde{U} + T_1(d\overline{T}_1).$$

$\tilde{U}$ is an analytic function of z_2, therefore by Remark 2 the first term remains analytic in z_2 (however we fix to order the variables $z_1, z_2, 1/|y_2|$); the second term is even independent of z_2, so A_2 is analytic in z_2.

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